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AN ESTIMATE FOR THE ENTROPY OF HAMILTONIAN FLOWS

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ABSTRACT. In this paper, we present a generalization to Hamiltonian flows on symplectic manifolds of the estimate proved by Ballmann and Wojtkovski in [4] for the dynamical entropy of the geodesic flow on a compact Riemannian manifold of nonpositive sectional curvature. Given such a Riemannian manifold M, Ballmann and Wojtkovski proved that the dynamical entropy h_μ of the geodesic flow on M satisfies the inequality

$$h_\mu \geq \int\limits_{SM} \operatorname{Tr} \sqrt{-K(v)}\, d\mu(v),$$

where v is a unit vector in T_pM if p is a point in M, SM is the unit tangent bundle on M, $K(v)$ is defined as $K(v) = \mathcal{R}(\cdot, v)v$, where $\mathcal{R}$ is the Riemannian curvature of M, and μ is the normalized Liouville measure on SM.

We consider a symplectic manifold M of dimension $2n$, and a compact submanifold N of M, given by the regular level set of a Hamiltonian function on M; moreover, we consider a smooth Lagrangian distribution on N, and we assume that the reduced curvature $\hat{R}^h_z$ of the Hamiltonian vector field $\vec{h}$ with respect to the distribution is nonpositive. Then we prove that under these assumptions, the dynamical entropy h_μ of the Hamiltonian flow with respect to the normalized Liouville measure on N satisfies

$$h_\mu \geq \int\limits_{N} \operatorname{Tr} \sqrt{-\hat{R}^h_z}\, d\mu.$$

1. THE CURVATURE

Let M be a $2n$-dimensional smooth manifold endowed with the symplectic structure σ. Let $h : M \to \mathbb{R}$ be a smooth function on the manifold, $\vec{h}$ be the Hamiltonian vector field associated with it, and $d_z h = \sigma(\cdot, \vec{h}(z))$. Assume that $\vec{h}$ is a complete vector field; we will denote by $\phi^t(\cdot) := e^{t\vec{h}(\cdot)}(\cdot)$ the flow generated by $\vec{h}$. Let Λ be a Lagrangian distribution on M; for any

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$z \in M$, we define the bilinear mapping

$$g_z^h : \Lambda_z \times \Lambda_z \to \mathbb{R}, \quad g_z^h(X,Y) = \sigma([\vec{h}, X], Y), \quad X, Y \in \Lambda_z.$$

Definition 1. The Hamiltonian vector field $\vec{h}$ is said to be regular at $z \in M$ with respect to the Lagrange distribution Λ if the bilinear form g_z^h is nondegenerate. A regular Hamiltonian vector field $\vec{h}$ is said to be monotone at $z \in M$ with respect to Λ if the form g_z^h is sign-definite.

Example 1. Assume that Λ is an involutive Lagrangian distribution. Then, by the Darboux–Weinstein theorem, there exist local coordinates $\{(p,q) : p, q \in \mathbb{R}^n\}$ such that $\sigma = \sum_{i=1}^{n} dp_i \wedge dq^i$ and $\Lambda_z = \{(p,0)\}$; in these coordinates, the previous requirement about the bilinear form g_z^h is equivalent to the requirement that the matrix $\left\{\dfrac{\partial^2 h}{\partial p_i \partial p_j}\right\}$ is nondegenerate and sign-definite.

Assume that $\vec{h}$ is regular and monotone. We define a curve in the Lagrange Grassmannian $L(T_zM)$ setting $J_z(0) = \Lambda_z$ and $J_z(t) = \phi_*^{-t}\Lambda_{\phi^t z}$; this curve is called the *Jacobi curve*. Using the terminology of [2], the curve is regular since the bilinear form g_z^h is nondegenerate. We have that for any t sufficiently close (but not equal to) 0, $J_z(t)$ is transversal to $J_z(0)$ [1]. Denote by $\pi_{J_z(t)J_z(0)}$ the projector of T_zM on $J_z(0)$ and parallel to $J_z(t)$, and note that the space $\{\pi_{\Delta J_z(0)} : \Delta \in G_n(T_zM), \Delta \in J_z(0)^{\pitchfork}\}$ is an affine subspace of $\mathfrak{gl}(T_zM)$ (see [1]); if we compute the Laurent expansion around 0 of the operator-valued function $t \mapsto \pi_{J_z(t)J_z(0)}$, i.e., $\pi_{J_z(t)J_z(0)} = \pi_0 + \sum_{i \neq 0} t^i \pi_i$, we can prove that for $i \neq 0$, $\pi_i \in \mathfrak{gl}(T_zM)$, while π_0 is an element of the affine space and hence there exists a unique $\Delta \in J_z(0)^{\pitchfork}$ such that $\pi_0 = \pi_{\Delta J_z(0)}$; this subspace is called the derivative element to $J_z(0)$ and is denoted by $J_z^{\circ}(0)$. Analogously, we can apply the same procedure to construct the derivative element to $J_z(t)$ for $t \neq 0$, and hence we can define the *derivative curve* of the curve $J_z(t)$: $t \mapsto J_z^{\circ}(t)$; moreover, we have that $J_z^{\circ}(t) = \phi_*^{-t} J_{\phi^t z}^{\circ}(0)$.

Since the Jacobi curve is regular, its derivative curve is smooth and lies in the Lagrange Grassmannian of T_zM [1]. These two curves form a splitting (which is called the canonical splitting) of T_zM into two Lagrangian subspaces $T_zM = J_z(t) \oplus J_z^{\circ}(t)$.

Let Δ_0 and Δ_1 be two transversal subspaces in the Grassmannian $G_n(T_zM)$ and ξ_0 and ξ_1 be two tangent vectors to $G_n(T_zM)$, respectively, at the points Δ_0 and Δ_1. Let $\gamma_i(t)$, $i = 0, 1$, be two curves in $G_n(T_zM)$ such that $\gamma_i(0) = \Delta_i$ and $\dfrac{d}{dt}\gamma_i(t)\Big|_{t=0} = \xi_i$. Let us define the operator in

$\mathfrak{gl}(\Delta_1)$:

$$[\xi_0, \xi_1] := \frac{\partial^2}{\partial t \partial \tau} \pi_{\gamma_0(t)\gamma_1(0)} \pi_{\gamma_0(0)\gamma_1(\tau)} \Big|_{\Delta_1} \Big|_{t=\tau=0};$$

this operator depends only on ξ_0 and ξ_1.

Definition 2. The operator $R_{J_z}(t) \in \mathfrak{gl}(J_z(t))$ defined as

$$R_{J_z}(t) := [\dot{J}_z^\circ(t), \dot{J}_z(t)]$$

is called the (*generalized*) *curvature* of the curve $J_z(t)$ at the time t.

If we choose local coordinates on the Jacobi curve and its derivative curve setting

$$J_z(t) \simeq \{(x, S_t x) : x \in \mathbb{R}^n\}, \quad J_z^\circ(t) \simeq \{(x, S_t^\circ x) : x \in \mathbb{R}^n\},$$

where S_t and S_t° are matrices of dimension n, then the curvature is

$$R_{J_z}(t) = (S_t^\circ - S_t)^{-1} \dot{S}_t^\circ (S_t^\circ - S_t)^{-1} \dot{S}_t.$$

Definition 3. The operator $R_z^h \in \mathfrak{gl}(J_z(0))$ defined as

$$R_z^h := R_{J_z}(0)$$

is called the *curvature* of the Hamiltonian vector field $\vec{h}$ at the point $z \in M$.

Let $\Sigma_z = \ker(d_z h)/\operatorname{span}\{\vec{h}(z)\}$ and $\psi_z : T_z M \to T_z M/\operatorname{span}\{\vec{h}(z)\}$ be the canonical projection on the factor space; the space Σ_z inherits a symplectic structure given by the restriction of the form σ. Now we set

$$J_z^h(t) = \phi_*^{-t}\Big[\Lambda_{\phi^t z} \cap \ker(d_{\phi^t z} h) + \operatorname{span}\{\vec{h}(\phi^t z)\}\Big]$$

(it can be shown that, actually, $J_z^h(t) = J_z(t) \cap \ker(d_z h) + \operatorname{span}\{\vec{h}(z)\}$) and

$$\bar{J}_z(t) = J_z^h(t)/\operatorname{span}\{\vec{h}(z)\};$$

$\bar{J}_z(t)$ is actually a curve in the Lagrange Grassmannian $L(\Sigma_z)$. If this Jacobi curve is regular, then its curvature operator $R_{\bar{J}_z}(t)$ is well defined on $\bar{J}_z(t)$.

Definition 4. The operator $\hat{R}_{J_z^h}(t)$ on $J_z^h(t)$ defined as

$$\hat{R}_{J_z^h}(t) := \left(\psi\big|_{J_z(t)\cap\ker(d_z h)}\right)^{-1} \circ R_{\bar{J}_z}(t) \circ \psi$$

is called the *curvature operator* of the h-reduction J_z^h at the time t.

Definition 5. The operator $\hat{R}_z^h$ on $J_z^h(0)$ defined as

$$\hat{R}_z^h := \hat{R}_{J_z^h}(0)$$

is called the *reduced curvature* of the Hamiltonian vector field $\vec{h}$ at the point $z \in M$.

Example 2. Let $M = \mathbb{R}^n \times \mathbb{R}^n$ and $h(p,q) = \frac{1}{2}|p|^2 + U(q)$. Consider the Lagrangian distribution $\Lambda_{(p,q)} = (\mathbb{R}^n, 0)$ and define the Jacobi curve $J_{(p,q)}(t) = \phi_*^{-t}\Lambda_{\phi^t(p,q)}$. Then we have that the curvature is given by

$$R^h_{(p,q)} = \frac{\partial^2 U}{\partial q^2}, \quad \hat{R}^h_{(p,q)} = \frac{\partial^2 U}{\partial q^2} + \frac{3}{|p|^2}(\nabla_q U, 0) \otimes (\nabla_q U, 0)^T.$$

Example 3. Let M be an n-dimensional smooth manifold and $h : T^*M \to \mathbb{R}$ be such that the restriction $h|_{T^*_{\pi(z)}M}$ (where $\pi : T^*M \to M$ is the canonical projection) is a positive quadratic form and hence defines a Riemannian structure on M. Let $J_z(0) = T_z(T^*_{\pi(z)}M)$; then we have that $R^h_z X = \mathcal{R}(\bar{z}, \bar{X})\bar{z}$ for any $X \in T_z(T^*_{\pi(z)}M)$, $z \in T^*M$, where $\mathcal{R}$ is the Riemann curvature tensor, $\bar{z}$ is a vector in TM obtained from z by the action of the metric tensor, and X is identified with a linear form of $T^*_z M$ via the isomorphism between $T_z(T^*_{\pi(z)}M)$ and $T^*_{\pi(z)}M$. The curvature operator of the h-reduction J^h_z is the same, $\hat{R}^h_z = R^h_z$.

Example 4. Let M be as in the previous example and the Hamiltonian function h be the sum of the Hamiltonian function of the previous example and the function $U \circ \pi$, where U is a function on M; then

$$R^h_z X = \mathcal{R}(\bar{z}, \bar{X})\bar{z} + D_X(\nabla U),$$

$$\hat{R}^h_z X = R^h_z X + \frac{3\left\langle \nabla_{\pi(z)} U, X \right\rangle_h}{2(h(z) - U(\pi(z)))}(\nabla_{\pi(z)} U, 0)^T,$$

where we denote by $\langle \cdot, \cdot \rangle_h$ the inner product defined by the Riemannian structure given by h, and where D_X is the Riemannian covariant derivative along X.

2. Results

Let M be a $2n$-dimensional smooth manifold endowed with the symplectic structure σ and $h : M \to \mathbb{R}$ be a smooth function on the manifold. We restrict the consideration to a regular sublevel N of the Hamiltonian function h, which is then a codimension-one submanifold of M, and we require that this submanifold is compact. Moreover, we assume that the Hamiltonian function satisfies the regularity condition, which will be specified later. Now we consider the flow generated by the Hamiltonian vector field $\vec{h}(z)$; note that it preserves the level sets of the Hamiltonian, i.e., $h(\phi^t z) = h(z)$ for all t; we are interested in computing the dynamical entropy $h_\mu(\phi)$, where μ is the (normalized) Liouville measure restricted to the submanifold N; it is defined as $d\mu = \frac{1}{\mathcal{N}}\sigma \wedge \cdots \wedge \sigma \wedge \iota_X \sigma$, where σ is multiplied by itself $n-1$ times, $\iota_X \sigma = \sigma(X, \cdot)$, X is a vector field on a neighborhood of N such that $\langle dh, X \rangle = 1$ and $\mathcal{N} = \int_N \sigma \wedge \cdots \wedge \sigma \wedge \iota_X \sigma$; it can be proved that this definition does not depend on the particular choice of such a vector field.

In order to compute the dynamical entropy, we are going to use the Pesin theorem [5], which states that the entropy is equal to the integral of the sum of positive Lyapunov exponents taken with their multiplicities, and hence we will compute the exponents of the Hamiltonian flow. Recall that the Lyapunov exponent in the point $z \in N$ along the direction $X \in T_zN$ is defined by

$$\lambda^{\pm}(z, X) = \lim_{t\to\pm\infty} \frac{1}{|t|} \log \|\phi^t_* X\|, \tag{1}$$

where $\|\cdot\|$ is the inner product defined on T_zN and, since N is compact, this definition does not depend on the choice of the norm. The symplectic form restricted to N has a one-dimensional kernel given by the span of the Hamiltonian vector associated with h; indeed

$$\sigma(v, \vec{h}) = \langle d_z h, v\rangle = 0 \quad \forall v \in T_zN$$

since $T_zN = \ker(d_z h)$; hence, for all $z \in N$, we can write $T_zN \simeq \Sigma_z \oplus \mathrm{span}\{\vec{h}(z)\}$, where $\Sigma_z = T_zN/\,\mathrm{span}\{\vec{h}(z)\}$ is a $(2n-2)$-dimensional vector space and the restriction $\bar{\sigma} = \sigma|_{\Sigma_z}$ induces a symplectic structure on Σ_z. Since $\mathrm{span}\{\vec{h}\}$ is preserved by the action of its flow, i.e., $\phi^t_*\vec{h}(z) = \vec{h}(\phi^t z)$, we can take the quotient and study the exponential divergence of the trajectories along directions given by vectors lying in Σ_z and, therefore, we will consider the mapping $\tilde{\phi}^t_* : \Sigma_z \to \Sigma_{\phi^t z}$, where $\tilde{\phi}^t_* = {\phi^t}_*|_{\Sigma_z}$.

Now we can state the result.

Theorem 1. *Let N be a compact, regular level set of a smooth Hamiltonian function defined on a smooth symplectic manifold of dimension $2n$, Λ be a Lagrangian distribution in $TN/\,\mathrm{span}\{\vec{h}\}$, and let the Hamiltonian vector field $\vec{h}$ be monotone on N with respect to Λ. Consider the Jacobi curve $\bar{J}_z(t) = \tilde{\phi}^{-t}_* \Lambda_{\phi^t z}$ and assume that the curvature $\hat{R}^h_z$ of $\vec{h}$ is nonpositive. Then the dynamical entropy h_μ of the Hamiltonian flow on N with respect to the normalized Liouville measure on N satisfies the inequality*

$$h_\mu \geq \int_N \mathrm{Tr}\sqrt{-\hat{R}^h_z}\, d\mu.$$

Proof. Owing to the sign-definiteness of the bilinear form g^h_z, we can endow Σ_z with an inner product; indeed, let us define (for positive-definite g^h_z) the following inner product on $\bar{J}_z(0)$:

$$\bar{J}_z(0) \ni X, Y \mapsto \langle X, Y\rangle'_h := \bar{\sigma}([\vec{h}, X], Y).$$

By means of the symplectic form, we can establish an isomorphism between $\bar{J}^\circ_z(0)$ and the dual of $\bar{J}_z(0)$: $\bar{J}^\circ_z(0) \ni W \mapsto \bar{\sigma}(W, \cdot) : \bar{J}_z(0) \to \mathbb{R}$, since there exists a unique $X_W \in \bar{J}_z(0)$ such that $\bar{\sigma}(W, \cdot) = \langle X_W, \cdot\rangle_h$. We can define the inner product on $\bar{J}^\circ_z(0)$ as follows:

$$\bar{J}^\circ_z(0) \ni W, V \mapsto \langle W, U\rangle^\circ_h := \langle X_W, X_V\rangle_h\,.$$

Now it is possible to define an inner product on the whole Σ_z: for any $X, Y \in \Sigma_z$, we set

$$\langle X, Y\rangle_h := \left\langle \pi_{\bar{J}^\circ_z(0)\bar{J}_z(0)}X, \pi_{\bar{J}^\circ_z(0)\bar{J}_z(0)}Y\right\rangle'_h + \left\langle \pi_{\bar{J}_z(0)\bar{J}^\circ_z(0)}X, \pi_{\bar{J}_z(0)\bar{J}^\circ_z(0)}Y\right\rangle^\circ_h;$$

by definition, $\bar{J}^\circ_z(0)$ is orthogonal to $\bar{J}_z(0)$ with respect to the inner product just defined.

Since the space Σ_z has a symplectic structure and the pair $(\bar{J}_z(t), \bar{J}^\circ_z(t))$ for any t forms a splitting of Lagrangian subspaces, given a basis $\{\epsilon^1, \dots, \epsilon^{n-1}\}$ of $\bar{J}_z(0)$, there is a unique way to choose a basis $\{e^1_z(t), \dots, e^{n-1}_z(t)\}$ of $J_z(t)$ such that $e^i_z(0) = \epsilon^i$ for all $i = 1, \dots, n$, $\{\dot{e}^1_z(t), \dots, \dot{e}^{n-1}_z(t)\}$ is a basis for $J^\circ_z(t)$ and $\{e^i_z(t), \dot{e}^i_z(t)\}_{i=1}^{n-1}$ is a Darboux basis for Σ_z, and it is called the *canonical moving frame* [1]. Moreover, as is shown in [1], the vectors $\ddot{e}^i_z(t)$, $i = 1, \dots, n-1$, lie in $\bar{J}_z(t)$ and

$$\ddot{e}^i_z(t) = \sum_{j=1}^{n-1} (-R_z(t))_{ij} e^j_z(t),$$

where $R_z(t)$ is the representation of the curvature $\hat{R}^h_z$ with respect to the basis $\{e^i_z(t)\}_{i=1}^{n-1}$, and it is symmetric.

For any $z \in N$, let us define the basis $\varepsilon_1(z), \dots, \varepsilon_{2n-2}(z)$ of Σ_z by setting $\varepsilon_i(z) = e^i_z(0)$, $\varepsilon_{i-n+1}(z) = \dot{e}^i_z(0)$, $i = 1, \dots, n-1$. Obviously, this basis is orthonormal for any z. Consider the vector

$$X = \sum_{i=1}^{2n-2} x_i\, \varepsilon_i(z) = \sum_{i=1}^{n-1} \eta_i(t)\, e^i_z(t) + \xi_i(t)\, \dot{e}^i_z(t) \in \Sigma_z, \tag{2}$$

where $(\eta_i(t), \xi_i(t))$ are the components of the vector with respect to the canonical moving frame and, obviously, $(\eta(0), \xi(0)) = (x_1, \dots, x_{2n-2})$. We can prove by a direct computation that the pair $(\eta(t), \xi(t))$ satisfies the system of first-order differential equations

$$\dot{\xi}(t) = -\eta(t), \tag{3}$$

and hence the vector $\xi(t)$ satisfies the second-order differential equation

$$\ddot{\xi}(t) + R_z(t)\xi(t) = 0. \tag{4}$$

Since the canonical moving frame is defined so that

$$e^i_{\phi^t z}(0) = \tilde{\phi}^t_* e^i_z(t), \quad \dot{e}^i_{\phi^t z}(0) = \tilde{\phi}^t_* \dot{e}^i_z(t), \quad i = 1, \dots, n-1,$$

we have

$$e^i_z(t) = \tilde{\phi}^{-t}_* e_{\phi^t z}(0) = \tilde{\phi}^{-t}_* \varepsilon_i(\phi^t z), \quad i = 1, \dots, n-1,$$
$$\dot{e}^i_z(t) = \tilde{\phi}^{-t}_* \dot{e}_{\phi^t z}(0) = \tilde{\phi}^{-t}_* \varepsilon_i(\phi^t z), \quad i = n, \dots, 2n-2.$$

Hence

$$\tilde{\phi}^t_* X = \sum_{i=1}^{n-1} \eta_i(t)\, \tilde{\phi}^t_* e^i_z(t) + \xi_i(t)\, \tilde{\phi}^t_* \dot{e}^i_z(t)$$

$$= \sum_{i=1}^{n-1} \eta_i\,(t)\varepsilon_i(\phi^t z) + \xi_i(t)\, \epsilon_{i+n-1}(\phi^t z),$$

and this means that the components of $\tilde{\phi}^t_* X$ with respect to the basis $\{\varepsilon_i(\phi^t_z)\}_{i=1}^{2n-2}$ of $\Sigma_{\phi^t z}$ are the same as the components of X with respect to the canonical moving frame at time t. Since the basis $\{\varepsilon_i(z)\}_i$ is orthonormal for any z, we find that

$$\|\pi_{\bar{J}_{\phi^t z} \bar{J}^\circ_{\phi^t z}} \tilde{\phi}^t_* X\| = |\xi(t)|, \quad \|\pi_{\bar{J}^\circ_{\phi^t z} \bar{J}_{\phi^t z}} \tilde{\phi}^t_* X\| = |\dot{\xi}(t)|.$$

Now we compute the Lyapunov exponents on N; by the multiplicative ergodic theorem [5] we know that limit (1) exists a.e. (with respect to the standard Liouville measure normalized on N) in N. Hence we can define the following subspaces of Σ_z:

$$E^u_z = \{X \in \Sigma_z : \lambda^-(z, X) < 0\},$$
$$E^s_z = \{X \in \Sigma_z : \lambda^+(z, X) < 0\},$$
$$E^0_z = \{X \in \Sigma_z : \lambda^-(z, X) \le 0 \text{ and } \lambda^+(z, X) \le 0\};$$

these subspaces span Σ_z. For any subspace E_z of Σ_z such that $E^u_z \subset E_z \subset E^u_z \oplus E^0_z$, we have that

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \log|\det(\tilde{\phi}^t_*|_{E_z})| = \pm\chi(z),$$

where $\chi(z)$ is the sum of the positive Lyapunov exponents in z, taken with their multiplicities.

Knowing this, we are now looking for such a subspace E_z; we will see that a good candidate is the graph of a proper linear operator U_z defined from $\bar{J}^\circ_z(0)$ to $\bar{J}_z(0)$.

Now we introduce for any $z \in N$ the subset $H(z)$ of Σ_z such that

$$H(z) = \left\{ X \in \Sigma_z : \frac{d}{dt} \left\| \pi_{\bar{J}_{\phi^t z}(0) \bar{J}^\circ_{\phi^t z}(0)} \tilde{\phi}^t_* X \right\| \ge 0 \ \forall t \right\};$$

clearly, $H(z)$ is intrinsically defined and is invariant along the trajectory $\phi^t z$.

In the following, for simplicity, we will denote $\bar{J}_{\phi^t z}(0)$ by $v(t)$ and $\bar{J}^\circ_{\phi^t z}(0)$ by $v^\circ(t)$.

Lemma 1. *$H(z)$ is a subspace of Σ_z.*

Proof. From the convexity of $\|\pi_{v(t)v^\circ(t)} \tilde{\phi}^t_* X\|^2$ (see (4)) we deduce that a vector $X \in \Sigma_z$ belongs to $H(z)$ if and only if $\|\pi_{v(t)v^\circ(t)} \tilde{\phi}^t_* X\|$ is bounded for negative times. Linear combinations of vectors having this property satisfy this requirement. □

Lemma 2. $H(z) \cap v(0) = \{0\}$.

Proof. A vector $X \in H(z)$ belongs to $v(0)$ if $\pi_{v(0)v^\circ(0)}X = 0$, i.e., if $\xi(0) = 0$. On the contrary, suppose that such a (nonzero) vector is contained in $H(z)$. Then

$$\frac{d^2}{dt^2}|\xi(t)|^2\Big|_{t=0} = \left\langle \dot{\xi}(0), \dot{\xi}(0) \right\rangle - \langle R_z(t)\xi(0), \xi(0)\rangle > 0,$$

hence 0 is a strong minimum for $|\xi(t)|$, which contradicts the definition of $H(z)$. □

Lemma 3. *$H(z)$ is a Lagrangian subspace.*

Proof. For any $\tau \in \mathbb{R}$, we define

$$H_\tau = \left\{ X \in \Sigma_z : \frac{d}{dt}\|\pi_{v(t)v^\circ(t)}\tilde{\phi}^t_* X\| \geq 0 \ \forall t \geq \tau \right\};$$

we have that $H_{\tau_1} \subseteq H_{\tau_2}$ for $\tau_1 \leq \tau_2$ and that $H(z) = \bigcap_\tau H_\tau$.

The set H_τ contains a Lagrangian subspace for any τ. Indeed, fix τ and consider

$$V_\tau = \{X \in \Sigma_z : \pi_{v(0)v^\circ(t)}\tilde{\phi}^\tau_* X = 0\}.$$

Using coordinates, we prove that this subspace is contained in H_τ : if we write

$$X = \sum_{i=1}^{n-1} -\dot{\xi}(t)e^i_z(t) + \xi(t)\dot{e}^i_x(t),$$

we have that

$$\|\pi_{v(t)v^\circ(t)}\tilde{\phi}^t_* X\| = |\xi(t)|$$

and hence, since

$$\frac{d}{dt}|\xi(t)|^2|_{t=\tau} = 0, \quad \frac{d^2}{dt^2}|\xi(t)|^2 \geq 0 \quad \forall t,$$

we see that

$$\frac{d}{dt}\|\pi_{v(t)v^\circ(t)}\tilde{\phi}^t_* X\| \geq 0$$

for all $t \geq \tau$ and $V_\tau \subset H_\tau$.

Now, since $\tilde{\phi}^\tau_* V_\tau = \bar{J}_{\phi^\tau z}(0)$, and this last subspace is Lagrangian, we proved our claim. $H(z)$ contains also a Lagrangian subspace; indeed, let us define for any τ the set $\hat{H}_\tau = \{V \in L(\Sigma_z) : V \subset H_\tau\}$, which is a compact nonempty subset in the Lagrange Grassmannian $L(\Sigma_z)$. Moreover, since $\hat{H}_{\tau_1} \subseteq \hat{H}_{\tau_2}$ for $\tau_1 \leq \tau_1$, we see that $\bigcap_\tau \hat{H}_\tau \neq \varnothing$; hence, since $\hat{H}_\tau \subset H_\tau$ for any τ, we can conclude that $H(z) \supseteq \bigcap_\tau \hat{H}_\tau \neq \varnothing$, which means that $H(z)$ contains a Lagrangian subspace.

From Lemma 2 we know that $\dim H(z) \le n-1$ and hence we can conclude that $H(z)$ is a Lagrangian subspace. □

Since the space $H(z)$ is Lagrangian and $H(z) \cap \bar{J}_z(0) = 0$ for all z, there exists a symmetric linear operator $U_z : \bar{J}^\circ_z(0) \to \bar{J}_z(0)$ such that for any element $X \in H(z)$ we have that $X = x + U_z(0)x$, where $x \in \bar{J}^\circ_z(0)$, i.e., $H(z)$ is the graph of the operator U_z.

Hence we can find a linear operator $V_z : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that if

$$H(z) \ni X = \sum_{i=1}^{n-1} \eta_i(0)\varepsilon_i(z) + \xi(0)\varepsilon_{i+n-1}(z),$$

then $\eta(0) = -V_z\xi(0)$, and, by (3), we obtain that $\dot{\xi}(0) = V_z\xi(0)$, by (4) that the operator satisfies the equation

$$\dot{V}_{\phi^t z} + V^2_{\phi^t z} + R_z(t) = 0. \tag{5}$$

By the definition of $H(z)$, the operator V_z is nonnegative definite for any z.

Lemma 4. $E^u_z \subset H(z) \subset E^u_z \oplus E^0_z$.

Proof. Let $X \in E^u(z)$ and $Y \in E^u_z \oplus E^0_z$;

$$\begin{aligned}\lim_{t\to-\infty} & \frac{1}{|t|} \log |\bar{\sigma}(\tilde{\phi}^t_* X, \tilde{\phi}^t_* Y)| \\ &\le \lim_{t\to-\infty} \left[\frac{1}{|t|}\log\|\bar{\sigma}\| + \frac{1}{|t|}\log\|\tilde{\phi}^t_* X\| + \frac{1}{|t|}\log\|\tilde{\phi}^t_* Y\|\right] \\ &\qquad = \lambda^-(z,X) + \lambda^-(z,Y) < 0,\end{aligned}$$

and this implies that $\bar{\sigma}(\tilde{\phi}^t_* X, \tilde{\phi}^t_* Y) \to 0$ as $t \to -\infty$. Since $\tilde{\phi}^{t*}\bar{\sigma} = \bar{\sigma}$, we obtain that $\bar{\sigma}(X,Y) = 0$ and hence E^u_z and $E^u_z \oplus E^0_z$ are skew-orthogonal. By a dimensional computation, we can prove that actually E^u_z and $E^u_z \oplus E^0_z$ are skew-orthogonal complements to each other.

Let $X \in E^u_z$, i.e.,

$$\lim_{t\to-\infty} \frac{1}{|t|} \log \|\tilde{\phi}^t_* X\| < 0,$$

and this means that $\|\tilde{\phi}^t_* X\| < 1$, which implies that $\tilde{\phi}^t_* X$ is bounded in norm for nonpositive times, and, consequently, $\pi_{v(t)v^\circ(t)}\tilde{\phi}^t_* X$ is also bounded in norm, which implies that

$$\tilde{\phi}^t_* X \in H(\phi^t z) = \tilde{\phi}^t_*[H(z)] \quad\Rightarrow\quad E^u_z(z) \subset H(z).$$

Since $H(z)$ is Lagrangian, we also find that $H(z) \subset E^u_z(z) \oplus E^0_z$. □

Lemma 5. *Let $X \in H(z)$; then $\pi_{v(0)v^\circ(0)}X \in \ker U_z$ if and only if*

$$\|\pi_{v^\circ(t)v(t)}\tilde{\phi}^t_* X\| = 0$$

for any $t \le 0$.

Proof. We prove the assertion in the coordinates. Let X be as in (2) such that $\xi(0) \in \ker V_z$, i.e., $\eta(0) = 0$. Since by convexity (4)

$$\frac{d^2}{dt^2}|\xi(t)|^2 \geq 0,$$

and by the hypothesis

$$\frac{d}{dt}|\xi(t)|^2|_{t=0} = 0,$$

we obtain that $|\xi(t)|^2$ remains constant for all $t \leq 0$, which implies, using again the convexity, that $|\dot{\xi}(t)| = 0$ for all $t \leq 0$. Conversely, if $\dot{\xi}(t) = 0$ for all $t \leq 0$, then obviously we obtain the thesis. □

Denote by $H_0(z)$ the graph of U_z restricted to the orthogonal complement in $\bar{J}_z^\circ(0)$ to $\ker U_z$; it follows from the above lemma that $\tilde{\phi}_*^t[H_0(z)] \subseteq H_0(\phi^t z)$ for all $t \geq 0$. Indeed, let $X \in H(z)$ be such that $\xi(0) \in \ker V_z$; then, by the previous results, $\dot{\xi}(t) = 0$ for any $t \leq 0$, which means that $\xi(t) \in \ker V_{\phi^t z}$ for negative times, i.e., $\pi_{v^\circ(t)v(t)}\tilde{\phi}_*^t X \in \ker U_{\phi^t z}$.

Since the dimension of $H_0(z)$ is nondecreasing along the orbits of the Hamiltonian flow, we obtain that $\dim H_0(z)$ is constant on a ϕ^t-invariant set of the full measure, and hence $\phi_*^t[H_0(z)] = H_0(\phi^t z)$ on this set.

We will work in the space $H_0(z)$ since we need the operator U_z to be strictly positive definite, and we denote by U_z^0 the restriction of U_z to the orthogonal complement to $\ker U_z$ in $\bar{J}_z^\circ(0)$, and, respectively, V_z^0 and $R_z^0(t)$ are the restrictions of V_z and $R_z(t)$ to the orthogonal complement of $\ker V_z$ in $\mathbb{R}^{n-1}$; to do this, we will prove that actually it satisfies Lemma 4. First, we need the following result.

Lemma 6. *$R_z(t)$ vanishes on $\ker V_{\phi^t z}$ and both $R_z(t)$ and $V_{\phi^t z}$ preserve the orthogonal complement in $\mathbb{R}^{n-1}$ to $\ker V_{\phi^t z}$.*

Proof. Denote by $\Delta_z(t)$ the orthogonal complement in $\mathbb{R}^{n-1}$ to $\ker V_{\phi^t z}$. Let $X \in H(z)$, $(-\dot{\xi}(t), \xi(t))$ be its components as in (2), and let $\xi(t) \in \ker V_{\phi^t z}$. Then, by the previous lemma, $\dot{\xi}(\tau) = 0$ for $\tau \leq t$, which implies also vanishing the second derivative, i.e., $R_z(t)\xi(t) = 0$. Now let $x \in \ker V_{\phi^t z}, x' \in \Delta_z(t)$; since $\langle x, R_z(t)x' \rangle = \langle R_z(t)x, x' \rangle = 0$, we conclude that $R_z(t)[\Delta_z(t)] \subseteq \Delta_z(t)$. In the same way, we can show that $V_{\phi^t z}[\Delta_z(t)] \subseteq \Delta_z(t)$. □

Let $X \in H(z) \setminus H_0(z)$; then $\tilde{\phi}_*^t X$ is constant in norm with respect to t for any nonpositive t. Hence $\lambda^-(z, X) = 0 \Rightarrow X \notin E_z^u$, which implies that $E_z^u \subset H_0(z)$. Moreover, consider $X = X^{(1)} + X^{(2)} \in H(z)$; as usual, $(-\dot{\xi}^{(i)}(t), \xi^{(i)}(t))$ are the components of $\tilde{\phi}_*^t X^{(i)}$ with respect to the orthonormal frame $\{\varepsilon_i(\phi^t z)\}_i$, and we assume that $\xi^{(1)}(t) \in \ker V(z)$ and $\xi^{(2)}(t)$ lies in $\Delta_z(t)$ (defined as above). By the previous results, we obtain that $\dot{\xi}^{(1)}(t) = 0$ for $t \leq 0$, and hence $\ddot{\xi}^{(1)}(t) = 0$ for $t \leq 0$, and also $R_z(t)\xi^{(1)}(t) = 0$, which implies that both $\xi^{(1)}$ and $\xi^{(2)}$ satisfy Eq. (4).

Since $H_0(z)$ is the graph of the operator U_z^0, we can express the inner product on $H_0(z)$ in terms of the inner product on $\mathbb{R}^{n-1}$, setting

$$\langle X, Y \rangle_h = \left\langle \xi^X(0), A_z(0)\xi^Y(0) \right\rangle_c ,$$

where $A_z(t) = \mathbb{I} + {V_{\phi^t z}^0}^2$ (X and Y are as above) and $\langle \cdot, \cdot \rangle_c$ denotes the canonical inner product on $\mathbb{R}^{n-1}$.

We denote by $a_z(t) = |\det \phi_*^t|_{H_0(z)}|$ the determinant of ϕ_*^t with respect to the inner product defined by $A_z(t)$; hence we have that

$$a_z(t) = \sqrt{\det A_z(t)}|\det \phi_*^t|_{H_0(z)}|_c = \sqrt{\det A_z(t)}|\det e^{\int_0^t V_{\phi^s z} ds}|_{H_0(z)}|_c.$$

We define

$$r_z(t) := \frac{d}{dt} \log a_z(t) = \frac{1}{2} \operatorname{Tr} \dot{A}_z(t) A_z^{-1}(t) + \operatorname{Tr} V_{\phi^t z}^0,$$

and we obtain by a direct computation that

$$r_z(t) = \operatorname{Tr} \left[(V_{\phi^t z}^0 - R_z^0(t) V_{\phi^t z}^0)(\mathbb{I} + {V_{\phi^t z}^0}^2)^{-1} \right].$$

Since

$$\chi(z) = \lim_{t\to\infty} \frac{1}{t} \log|\det(\phi_*^t z|_{H_0(z)})| = \lim_{t\to\infty} \frac{1}{t} \log a_z(t) = \lim_{t\to\infty} \frac{1}{t} \int_0^t r_z(s)\, ds,$$

by the Birkhoff ergodic theorem [5] we obtain that, provided that r_z is an integrable function on N,

$$h_\mu(\phi) = \int_N \chi(z)\, d\mu(z) = \int_N r_z(0)\, d\mu.$$

Now we compute the dynamical entropy using a different inner product on $H_0(z)$, after showing that we will obtain the same value. Denote $A'_z(t) = V_{\phi^t z}^0$, and define the inner product

$$\langle X, Y \rangle' = \left\langle \xi^X(0), A'_z(0)\xi^Y(0) \right\rangle ;$$

we also obtain that

$$r'_z(t) = \frac{1}{2} \operatorname{Tr} \left[V_{\phi^t z}^0 - R_z^0(t) {V_z^0}^{-1} \right].$$

The volume element on N with respect to the inner product given by A' is related to the standard volume element as follows:

$$d\mu' = \sqrt{\frac{\det A'}{\det A}} d\mu.$$

If we set

$$c(t) = \frac{d\mu}{d\mu'} = \sqrt{\frac{\det A(t)}{\det A'(t)}} > 1,$$

then we find that $0 < a'(t) < a(t)c(0)$. Then we have

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t r_z'(s)\, ds = \limsup_{t\to\infty} \frac{1}{t} \log a_z'(t) \le \lim_{t\to\infty} \frac{1}{t} \log a_z(t) = \chi(z),$$

$$\liminf_{t\to-\infty} \frac{1}{|t|} \int_t^0 r_z'(s)\, ds = -\limsup_{t\to-\infty} \frac{1}{|t|} \log a_z'(t) \ge -\lim_{t\to-\infty} \frac{1}{|t|} \log a_z(t) = \chi(z),$$

and hence

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t r_z'(s)\, ds \le \chi(z) \le \liminf_{t\to-\infty} \frac{1}{|t|} \int_t^0 r_z'(s)\, ds;$$

r_z' is measurable on N since it is continuous. Applying the following lemma (see [4]), we can prove that it is also integrable on N.

Lemma 7. *Let ϕ^t be a measure-preserving flow on a probability space (X, μ) and $f : X \to \mathbb{R}$ be a measurable nonnegative function. If*

$$\limsup_{T\to+\infty} \frac{1}{T} \int_0^T f(\phi^t x)\, dt \le k(x)$$

for a.e. $x \in X$, where $k : X \to \mathbb{R}$ is a measurable function, then

$$\int_X f(x)\, d\mu(x) \le \int_X k(x)\, d\mu(x).$$

Hence we obtain by the ergodic theorem and equality of time averages in the future and in the past that

$$\int_N r_z(0)' d\mu = \int_N \chi(z)\, d\mu(z) = h_\mu(\phi).$$

Finally, we use the following result (see [4]).

Lemma 8. *Given three symmetric linear operators U, M, and N on a Euclidean space such that M and N are nonnegative definite and U is strictly positive definite, we obtain that*

$$\mathrm{Tr}[MU + NU^{-1}] \ge 2\,\mathrm{Tr}\,\sqrt{M}\sqrt{N},$$

where the equality holds if and only if $\sqrt{M}U = \sqrt{N}$.

Since we have that

$$r_z'(t) = \frac{1}{2}\,\mathrm{Tr}\left[V^0_{\phi^t z} - R^0_z(t) {V^0_{\phi^t z}}^{-1}\right],$$

where $V^0_{\phi^t z}$ is (strictly) positive definite and $-R^0_z(t)$ is nonnegative definite, we can apply the previous lemma with $U = V^0_{\phi^t z}$, $M = \mathbb{I}$, and $N = -R^0_z(t)$, obtaining

$$\frac{1}{2}\operatorname{Tr}[V^0_{\phi^t z} - R^0_z(t){V^0_{\phi^t z}}^{-1}] \geq \operatorname{Tr}\sqrt{-R^0_z(t)},$$

and hence

$$h_\mu(\phi) \geq \int\limits_N \operatorname{Tr}\sqrt{-R^0_z(0)}\,d\mu = \int\limits_N \operatorname{Tr}\sqrt{-R_z(0)}\,d\mu.$$

The lemma is proved. □

Remark. The estimate is sharp (i.e., we have the equality) if and only if $V^0_{\phi^t z} = \sqrt{-R^0_z(t)}$ for almost all $z \in N$, which implies that $V^2_{\phi^t z} = -R_z(t)$ almost everywhere on N, and hence, by the continuity, for every $z \in N$; this means that $\dot{V}_{\phi^t z} = 0$ on N, i.e., all Jacobi curves are symmetric [1].

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