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ON THE CODIMENSION ONE MOTION PLANNING PROBLEM

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ABSTRACT. In this paper, we improve the results of [5] related to motion planning problems for corank one sub-Riemannian (SR) metrics. First, we give the exact estimate of the metric complexity, in the generic 3-dimensional case. (Only bounds from above and from below were given in [5].) Second, we show that the general expression for the metric complexity (that was proven to hold generically in the C^∞ case, or under certain nonvanishing condition (C) in the analytic case) is, in fact, always true under condition (C), on the complement of a subset of codimension infinity, in the set of C^∞ "motion planning problems." Both results are constructive, i.e., an "asymptotic optimal synthesis" is exhibited in both cases.

1. Introduction

1.1. **Practical motivation.** There are so many possible references for the questions we discuss in this section that it is not realistic to give an exhaustive list of them. We have chosen a single one: [11], but this choice is completely arbitrary.

For us, a robot is a device described by a kinematic motion with nonholonomic constraints (linear constraints in our case). Engineers in robotics consider the "motion planning problem": given a source and a target, find an "admissible path" for the robot, connecting the source and the target and satisfying further requirements, e.g., avoiding certain obstacles, etc.

Moreover, the path to be found is required to have a "reasonable cost," the cost being usually specified by a quadratic form on the space of admissible velocities. Under very weak nonintegrability assumptions, such a cost defines a distance d, called the sub-Riemannian, or Carnot–Carathéodory distance.

The problem is usually solved in two steps.

Step 1: the purely geometric problem of finding a (nonadmissible) path Γ satisfying the additional requirements;

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Step 2: approximating this nonadmissible path by an admissible path γ.

In this paper, we address only the second step of the problem, i.e., we assume that the nonadmissible path Γ to be approximated by the nonholonomic motion is already chosen.

Whenever the "cost" is specified as the Carnot–Carathéodory distance d, one may define, following, e.g., Jean [7–9] the "metric complexity" of the given nonadmissible path Γ as follows: find the minimum length $l(\gamma_\varepsilon)$ of an admissible path γ_ε connecting the source and the target, and remaining at distance at most ε from the given Γ. Then the complexity is the asymptotic equivalent $MC(\varepsilon)$ (as ε tends to zero) of $l(\gamma_\varepsilon)/\varepsilon$.

Another crucial notion is the notion of an "asymptotic optimal synthesis": it is an explicit ε-dependent family of admissible curves γ_ε that realizes an equivalent of the metric complexity as ε tends to zero. Finding such an asymptotic optimal synthesis is equivalent, roughly speaking, to solving the motion planning problem "in practice."

1.2. **Contents of the paper.** The purpose of this paper, after [5], is to solve for *corank one* sub-Riemannian metrics (step 2 of) the motion planning problem in a *constructive way*, i.e., computing metric complexity and finding asymptotic optimal syntheses.

In [5], this was done for *generic* metrics of corank one in a space of dimension more than 3, and almost done in dimension 3. We recall these results (Theorems 1 and 2) in the next section of this introduction.

Our main results in this paper are as follows.

1. For the 3-dimensional case, we will give in Theorem 3 an explicit exact expression of the metric complexity together with a corresponding asymptotic optimal synthesis. Proofs of these results are given in Sec. 3.

2. For other dimensions, Theorem 4 proved in Sec. 4 contains much more than the generic statement of Theorem 1: it solves the motion planning problem in the C^∞ category, outside of a set of codimension infinity.

Section 2 is devoted to the precise statement and comments of these results.

For the purpose of proving these results, we will give (Sec. 4) some other complements to the paper [5], related to normal forms that are crucial for motion planning problems. These normal forms are important by themselves. For instance, they were used in studying sub-Riemannian balls of small radius in [1,3].

1.3. **Definitions and preliminary results.** For us, as we explained in Sec. 1.1, a "motion planning problem" over a smooth n-dimensional manifold Ξ is a triple $\Sigma = (\Delta, g, \Gamma)$ formed by a sub-Riemannian metric (Δ, g) over Ξ and $\Gamma : [0,1] \to \Xi$, a smooth compact curve, well parametrized, i.e., $\dfrac{d\Gamma(t)}{dt} \neq 0$ for all t without double point. Here, Δ is a codimension one distribution on Ξ, completely nonintegrable, g is a Riemannian metric over Δ.

If the curve Γ is everywhere transversal to Δ, the motion planning problem has been called *relevant* in [5]. If a motion planning problem is not relevant at a point $t_0 \in [0,1]$, then it is easy to deal with it in a neighborhood of t_0. The set of smooth (respectively, analytic) relevant motion planning problems is denoted by $\mathcal{S}^\infty$ (respectively, $\mathcal{S}^\omega$), and we endow it with the topology of C^∞ uniform convergence on compact sets. In the whole paper, we restrict the consideration to this set.

Given two functions f_1 and f_2 of a small parameter $\varepsilon > 0$ tending to $+\infty$ as ε tends to zero, let us say that f_1 is "strongly equivalent" to f_2, $f_1 \eqsim_s f_2$ (respectively, f_1 is "weakly equivalent" to f_2, $f_1 \eqsim_w f_2$) if

$$\lim_{\varepsilon \to 0} \frac{f_1(\varepsilon)}{f_2(\varepsilon)} = 1$$

(respectively, $k_1 f_1(\varepsilon) \leq f_2(\varepsilon) \leq k_2 f_1(\varepsilon)$, for sufficiently small ε and certain constants $k_1, k_2 > 0$).

Let d denote the sub-Riemannian distance over Ξ and let $\mathcal{T}_\varepsilon$ denote the ε-sub-Riemannian tube around Γ: $\mathcal{T}_\varepsilon = \{x \in \Xi \mid d(x,\Gamma) \leq \varepsilon\}$.

The fundamental "motion planning problem" is as follows: find an admissible (i.e., almost everywhere tangent to Δ) Lipschitz curve $\gamma : [0, T_\gamma] \to \mathcal{T}_\varepsilon$, $\gamma(0) = \Gamma(0)$, $\gamma(T_\gamma) = \Gamma(1)$, with minimum SR length. We may take γ parametrized by the arc length, so that the SR length is the time $T\gamma$, and $T\gamma$ is minimum possible. Let $T^*(\varepsilon)$ denote this minimum value.[1]

The class of strong (respectively, weak) equivalence of the function $MC_\Sigma : \varepsilon \to \frac{1}{\varepsilon} T^*(\varepsilon)$ is called the strong (respectively, weak) *metric complexity* of the problem Σ. This notion has been introduced by Jean [7–9].

In the same way, for $T < 1$, we denote by $MC_\Sigma(\varepsilon, T)$ the metric complexity of the piece $\Gamma([0,T])$ of Γ.

Let us now define what we mean by a (weak or strong) *asymptotic optimal synthesis*: it is an (ε-dependent) control strategy γ_ε that realizes a (weak or strong) equivalent of the metric complexity as ε tends to zero, i.e., it is a family $\{\gamma_\varepsilon\}$ of admissible curves, $\gamma_\varepsilon([0,\theta_\varepsilon]) \subset \mathcal{T}_\varepsilon$, $\gamma_\varepsilon(0) = \Gamma(0), \gamma_\varepsilon(\theta_\varepsilon) = \Gamma(1)$, such that $\frac{1}{\varepsilon} SR\text{-length}(\gamma_\varepsilon) \eqsim MC_\Sigma(\varepsilon)$.

In order to state our results, we need the notion of the *fundamental* 2-*form* α associated with a motion planning problem $\Sigma = (\Delta, g, \Gamma)$. Let ω be any one-form such that $\Delta = \mathrm{Ker}(\omega)$ and $\omega(d\Gamma/dt) = 1 \ \forall t \in [0,1]$. Then $\alpha = d\omega$ is uniquely defined along Γ, and is called the fundamental 2-form associated

[1] Recall that: (a) the nonintegrability assumption warrants the existence of admissible curves joining any couple of arbitrary points (the Rachevski–Chow theorem); (b) the sub-Riemannian length of an admissible curve is just the length measured via the metric g; (c) the sub-Riemannian distance between two points is the infimum of the SR length of admissible curves joining the two points.

with Σ. This 2-form α defines an associated *field of linear endomorphisms* $A(t)$ of $\Delta(\Gamma(t))$ by $g(A(t)X, Y) = \alpha(t)(X, Y)$ for all $X, Y \in \Delta(\Gamma(t))$.

We set

$$\kappa(t) = \sup |\text{eigenvalues of } A(t)|.$$

In [5], the following results have been proved.

Theorem 1 ($n \geq 4$)**.** *For* (a) Σ *in a generic* (*open, dense*) *subset of* $\mathcal{S}^\infty$*, or* (b) Σ *in* $\mathcal{S}^\omega$ *with the additional assumption* (c) $\alpha(t) \neq 0$ $\forall t \in [0, 1]$,

$$MC_\Sigma(\varepsilon, T) \eqsim_s \frac{2}{\varepsilon^2} \int_0^T \frac{1}{\kappa(t)} dt, \tag{1.1}$$

and the function

$$T \to \int_0^T \frac{1}{\kappa(t)} dt$$

is C^∞*-smooth in the case* (a) *and* C^1*-smooth and piecewise analytic in the case* (b)*.*

Remark 1. Note that formula (1.1) also makes sense if Σ is not relevant: in this case, the only thing that happens is that $\kappa(t)$ tends to ∞ at points where Γ is tangent to Δ. The integral is convergent and actually gives the right expression of the metric complexity.

Assume now that $n = 3$ and $\Gamma(t_i)$, $i = 1, \ldots, s$, are isolated Martinet points for Δ. Then $\kappa(t_i) = 0$ and $\kappa(t)$ is continuous and differentiable on the right and left side at $t = t_i$, and

$$\left| \frac{d\kappa}{dt}(t_i^+) \right| = \left| \frac{d\kappa}{dt}(t_i^-) \right| = \kappa_i.$$

The constants κ_i are invariants of Σ.

Theorem 2 ($n = 3$)**.** *For a generic* (*open dense*) *subset of* $\mathcal{S}^\infty$*, either* (1.1) *holds, or there is a finite set of times* $t_1, \ldots, t_m$*,* $t_i \in [0, T]$*, and associated positive constants* $c_1, \ldots, c_m$*, where* $\Gamma(t_i)$*,* $i = 1, \ldots, m$*, is a Martinet point for* Δ*, and*

$$MC_\Sigma(\varepsilon, T) \eqsim_s \sum_{i=1}^m -c_i \frac{\log \varepsilon}{\epsilon^2}. \tag{1.2}$$

Moreover, $\dfrac{4}{\kappa_i} \leq c_i$.

In both cases, these theorems are constructive: in the case of Theorem 1 (respectively, Theorem 2), a strong (respectively, weak) asymptotic optimal synthesis has been exhibited.

2. Statement of the results

We will show first the following final result, in the 3-dimensional case:

Theorem 3. *In Theorem 2, formula (1.2),* $c_i = \dfrac{4}{\kappa_i}$.

To state our result in arbitrary dimension, it will be easier to consider the curve Γ as fixed. Then, such a curve $\Gamma : [0,1] \to \Xi$ being given (well parametrized and without double points), we consider the set GS^∞ of germs along Γ of SR metrics (Δ, g) such that Δ is transversal to Γ. Let J^kSR^∞ be the bundle of k-jets of SR metrics over Ξ, and let J^kGS^∞ be the restriction of this bundle to Γ. Diffeomorphisms $\Phi : V(\Gamma(t_1)) \to V(\Gamma(t_2))$, mapping a neighborhood $V(\Gamma(t_1))$ of $\Gamma(t_1)$ in Ξ to a neighborhood $V(\Gamma(t_2))$ of $\Gamma(t_2)$ in Ξ, induce diffeomorphisms of the fibers of J^kGS^∞ over $\Gamma(t_1)$ and $\Gamma(t_2)$.

For each positive integer k, let $\mathcal{B}^k$ denote a subbundle (invariant with respect to the diffeomorphisms of the fibers just considered) of J^kGS^∞, the typical fiber of which is a stratified subset of the typical fiber of the ambient bundle J^kGS^∞, of codimension b_k in this fiber, and $b_k \to +\infty$ as $k \to +\infty$.

A subset $\mathbb{S}$ of GS^∞ is said to have infinite codimension (in GS^∞) if it consists of SR structures (Δ, g) such that at some point $t \in [0,1]$, the k-jets $j^k_{\Gamma(t)}(\Delta, g) \in \mathcal{B}^k$, for all k.

Theorem 4. *There is a subset* $\mathbb{S}$ *of* GS^∞ *of codimension* ∞ *in* GS^∞ *such that Theorem* 1 *still holds for* $(\Delta, g) \notin \mathbb{S}$ (*again under the additional assumption* (*C*) *that* $\alpha(t) \neq 0$ $\forall t \in [0,1]$).

In this case, the leading term

$$\frac{1}{\varepsilon^2}\int_0^T \frac{1}{\kappa(t)}dt$$

of $MC_\Sigma(\varepsilon, T)$ *is a* C^1 *smooth function of* T, *piecewise* C^∞.

In both cases, our proofs are constructive. The construction of the asymptotic optimal synthesis for Theorem 4 is exactly the same as for Theorem 1 in [5]. We do not re-explain it here.

Let us consider now the asymptotic optimal synthesis in the case of Theorem 3.

2.1. The asymptotic optimal synthesis in the Martinet case.

2.1.1. *Preliminaries.* We denote by $C_\varepsilon = \{x | d(x, \Gamma) = \varepsilon\}$ the boundary of the smooth tube $\mathcal{T}_\varepsilon$. We define a vector field X_ε *up to the sign* on C_ε, for sufficiently small ε.

If the distribution Δ is transversal to Γ, it is also transversal to C_ε provided that ε is small. Then we may define X_ε as follows:

$$X_\varepsilon(x) \in \Delta(x) \cap T_x C_\varepsilon, \quad \|X_\varepsilon(x)\|_g = 1.$$

We reparametrize Γ for $\Gamma : [-A, A] \to \Xi$, and $t = 0$ is a Martinet point. As was pointed out in [5], the vector field X_ε has a limit cycle near $\Gamma(0)$. This limit cycle is centered at a point $p_c = (0, 0, w_c)$ of Γ, of order ε^2 (i.e., $w_c \eqsim_w \varepsilon^2$) and its size has order ε^3 (in the generic situation).

To describe the asymptotic optimal synthesis, we need the "normal coordinates" and the normal form that have been introduced in [1].

2.1.2. *Normal coordinates and normal form.* It follows from [1] that the following two results hold, in the 3-dimensional case.

Proposition 1. (normal coordinates for a relevant motion planning problem)**.** *There is a neighborhood of* Γ *and coordinates* (x, y, w) *in this neighborhood with the following properties*:

(a) $\Gamma(t) = (0, 0, t)$, $\Delta(\Gamma(t)) = \mathrm{Ker}(dw)$, $g_{|\Gamma(t)} = dx^2 + dy^2$;
(b) *geodesics through* Γ *satisfying the Pontryagin maximum principle transversality conditions w.r.t.* Γ *are straight lines contained in the planes* $w = \mathrm{const}$;
(c) *the sub-Riemannian cylinder* C_ε *is equal to the Riemannian cylinder* $\{x^2 + y^2 = \varepsilon^2\}$.

These coordinates are unique modulo changes of coordinates of the form

$$(\tilde{x}, \tilde{y}) = \mathrm{Rot}_{\varphi(w)}(x, y), \tag{2.1}$$

where $\mathrm{Rot}_{\varphi(w)}$ *is the smooth rotation by the angle* $\varphi(w)$.

We associate cylindrical normal coordinates (ρ, θ, w) with (Euclidean) normal coordinates (x, y, w).

Theorem 5 (normal form for relevant motion planning problems)**.** *If a normal coordinate system* (x, y, w) *for* Σ *is given, then there exists a unique orthonormal frame* (F, G) *for* (Δ, g) *of the form*

$$\begin{aligned} F &= (1 + y^2)\beta\frac{\partial}{\partial x} - xy\beta\frac{\partial}{\partial y} + \frac{y}{2}\gamma\frac{\partial}{\partial w}, \\ G &= -xy\beta\frac{\partial}{\partial x} + (1 + x^2)\beta\frac{\partial}{\partial y} - \frac{x}{2}\gamma\frac{\partial}{\partial w}, \end{aligned} \tag{2.2}$$

where β *and* γ *are real smooth functions. Moreover, this normal form is invariant with respect to the changes of coordinates of type* (2.1) *from Proposition* 1.

Note that in this normal form, $\kappa(t) = |\gamma(0, 0, t)|$.

First, as was said above, we assume a single isolated Martinet point at $\Gamma(0)$, which means, in particular, that $0 = \kappa(0) = \gamma(0)$, and we replace the parametrization along Γ for $\gamma(0, 0, w)\frac{\partial}{\partial w}$ by $\alpha w\frac{\partial}{\partial w}$, where α is the invariant $\frac{d\kappa}{dt}(0^+)$. This is possible and does not affect the normal form

(2.2). Of course, we assume that $\alpha \neq 0$, which is a generic property for Σ (open dense for the C^∞ topology, in fact).

In cylindrical normal coordinates (ε, θ, w), it is easy to compute that the vector field X_ε can be written, up to the sign, as follows:

$$X_\varepsilon = \frac{1-\varepsilon^2\beta}{\varepsilon}\frac{\partial}{\partial\theta} + \frac{\varepsilon}{2}\Big(\alpha w + \varepsilon\cos\theta\tilde{\gamma}_1 + \varepsilon\sin\theta\tilde{\gamma}_2\Big)\frac{\partial}{\partial w}, \tag{2.3}$$

where $\tilde{\gamma}_i$ are smooth functions of $\varepsilon\cos\theta$, $\varepsilon\sin\theta$, w, $i = 1, 2$. We choose the sign of X_ε for $\alpha > 0$, and we eventually replace θ by $-\theta$, to keep $\dfrac{1-\varepsilon^2\beta}{\varepsilon}$ in expression (2.3). We choose a constant rotation in the (x, y) planes for $\tilde{\gamma}_1(0,0,0) = 0$. Then, $\tilde{\gamma}_2(0,0,0) = a$. Again, generically (open, dense in fact), $a \neq 0$, and up to a rotation by another angle π, we may assume $a > 0$.

It suffices to restrict the tube $\mathcal{T}_\varepsilon$ to some compact piece $\{|w| \leq W_0\}$ for some arbitrary $W_0 > 0$. If ε is sufficiently small, then the following properties hold on this (compact) tube $\mathcal{T}_\varepsilon$, for the trajectories of X_ε:

(a) $d\theta/dt > 0$;

(b)
$$\frac{dw}{d\theta} = \frac{\varepsilon^2}{2}\alpha w + \varepsilon^3 L(\varepsilon, \theta, w), \tag{2.4}$$
where $L(\varepsilon, \theta, w)$ is a smooth function such that $|L(\varepsilon, \theta, w)| < A$ for some constant $A > 0$;

(c) we set $K = 3A/\alpha$. For $|w| \leq \varepsilon K$,
$$\frac{dw}{d\theta} = \frac{\varepsilon^2}{2}\Big(\alpha w + \varepsilon a\sin\theta + \varepsilon^2 F(\varepsilon, \theta, w)\Big) \tag{2.5}$$
for some function F, smooth w.r.t. θ, w, and bounded independently of ε.

2.1.3. *The synthesis.* We give only one half of the asymptotic optimal synthesis, corresponding to the piece $w \geq 0$ of Γ. The piece $w \leq 0$ is similar (replacing w by $-w$ in the normal form). Also, we recall that by the results of [5], out of a neighborhood of the Martinet points, the asymptotic optimal synthesis is just given by any trajectory of X_ε.

Asymptotic optimal synthesis.

1. For $w \geq \varepsilon K$: follow any trajectory of X_ε (this strategy agrees with the asymptotic optimal synthesis away from Martinet points);
2. from $w_0 = 0$ to $w_1 = \varepsilon K$:
   (a) start from $\theta_0 = 0$, $k = 0$,
   (b) on the intervals $\theta \in [2k\pi, (2k+1)\pi]$ follow the flow of X_ε,
   (c) having reached $\theta = (2k+1)\pi$, cross the tube $\mathcal{T}_\varepsilon$ by a horizontal geodesic through Γ, contained in the plane $w_k = \text{const}$, to reach $\theta = (2k+2)\pi$.

3. The Martinet case

3.1. **Preliminaries.** In this section, we use the notation and results of Sec. 2.1.

We already know from [5] that the metric complexity at a generic Martinet point satisfies:

$$\frac{-4\log\epsilon}{\alpha\varepsilon^2} \leq MC_\Sigma(\varepsilon).$$

Let us just show that the metric complexity of the asymptotic synthesis described in Sec. 2.1.3 is strongly equivalent to $\frac{-4\log\epsilon}{\alpha\varepsilon^2}$. This will prove Theorem 3.

To do this, we consider a motion planning problem Σ in normal coordinates and in normal form (2.2) satisfying all the generic assumptions of Sec. 2.1.2 (i.e., $\alpha > 0$, $a > 0$). Again, $K = 3A/\alpha$ and we take $w_1 = \varepsilon K$. We fix $w_2 > w_1$ independently of ε.

We will show that:

(a) the time to go from a point $p_0 = (x_0, y_0, 0) \in C_\varepsilon$ to $p_1 = (x_1, y_1, w_1) \in C_\varepsilon$ is weakly equivalent to $1/\varepsilon$ (or less);

(b) the time to go from p_1 to a certain point p_2 of the form $p_2 = (x_2, y_2, w_2) \in C_\varepsilon$, is strongly equivalent to $\frac{-2\log\epsilon}{\alpha\varepsilon}$.

Taking into account the piece $w < 0$ of the curve Γ, which is subject to a similar treatment, this will prove the result. Some extra pieces of horizontal geodesics of length 2ε (respectively, ε) are needed at $w_0 = 0$ (respectively, at the endpoints of Γ), but their asymptotic cost is negligible.

3.2. **Proof of (b).** Trajectories of X_ε satisfy (2.4):

$$\frac{dw}{d\theta} = \frac{\varepsilon^2}{2}\alpha w + \varepsilon^3 L(\varepsilon, \theta, w).$$

The cylindrical coordinates of p_1 are $(\varepsilon, \theta_1, w_1)$. With an additional cost of 2ε, which is irrelevant for strong equivalence, we may assume that $\theta_1 = 0$ (following horizontal straight geodesics). Then, for $\theta \geq 0$, the following estimate holds on C_ε, if ε is small:

$$\begin{aligned} w(\theta) &\geq e^{\frac{\alpha\varepsilon^2}{2}\theta} w_1 - \varepsilon^3 A \int_0^\theta e^{\frac{\alpha\varepsilon^2}{2}(\theta - s)} ds \\ &\geq e^{\frac{\alpha\varepsilon^2}{2}\theta}\left(w_1 - \frac{2\varepsilon A}{\alpha}\right) + \frac{2\varepsilon A}{\alpha}. \end{aligned} \tag{3.1}$$

Since $w_1 = \varepsilon K$ and K can be taken as $3A/\alpha$, it follows that

$$\theta \leq -\frac{2\log\varepsilon}{\alpha\varepsilon^2} + 2\frac{\log\left(\frac{\alpha w(\theta)}{A}\right)}{\alpha\varepsilon^2}. \tag{3.2}$$

Now, along the trajectories of X_ε, θ satisfies

$$\frac{d\theta}{dt} = \frac{1-\varepsilon^2\beta}{\varepsilon} \geq \frac{1}{\varepsilon} - \varepsilon K_2,$$

for a certain $K_2 > 0$, and for sufficiently small ε. Assume that p_2 has cylindrical coordinates $(\varepsilon, \theta_2, w_2)$ and is reached at time T. Then

$$\theta_2 \geq \frac{T}{\varepsilon} - \varepsilon K_2 T$$

and $T \leq \theta_2\varepsilon(1+\varepsilon^2 K_3)$ for $K_3 > 0$. Using (3.2), we obtain:

$$T \leq \theta_2\varepsilon(1+\varepsilon^2K_3) \leq -\frac{2\log\varepsilon}{\alpha\varepsilon} + 2\frac{\log(\frac{\alpha w_2}{A})}{\alpha\varepsilon} + K_3'.$$

Since we already know by [5] that

$$T \geq -\frac{2\log\varepsilon}{\alpha\varepsilon} - \frac{K_4}{\varepsilon},$$

for $K_4 > 0$ (another straightforward estimate we did not recall), this proves the result.

3.3. **Proof of (a).** We consider a piece of trajectory of X_ε starting at time 0 from $p_k = (\epsilon, 2k\pi, w_k)$, and we denote by $p_{k+1} = (\epsilon, (2k+1)\pi, w_{k+1})$ the point reached when θ reaches $(2k+1)\pi$. Since

$$\frac{dw}{d\theta} = \frac{\varepsilon^2}{2}(\alpha w + \varepsilon a\sin(\theta) + \varepsilon^2 F(\varepsilon,\theta,w)),$$

we obtain

$$w_{k+1} \geq e^{\frac{\alpha\varepsilon^2}{2}\pi}w_k + \frac{a\varepsilon^3}{2}\int_{2k\pi}^{(2k+1)\pi} e^{\frac{\alpha\varepsilon^2}{2}((2k+1)\pi-s)}\sin(s)ds$$
$$- \frac{\varepsilon^4}{2}\int_{2k\pi}^{(2k+1)\pi} e^{\frac{\alpha\varepsilon^2}{2}((2k+1)\pi-s)}K_5 ds,$$

for some $K_5 > 0$. A straightforward computation shows:

$$w_{k+1} \geq e^{\frac{\alpha\varepsilon^2}{2}\pi}w_k + \frac{\varepsilon^3 a}{4}. \tag{3.3}$$

Applying this k times from $p_0 = (\epsilon, 0, 0)$ gives:

$$w_k \geq \frac{\varepsilon^3 a}{4}(1 + e^{\frac{\alpha\varepsilon^2}{2}\pi} + \cdots + e^{(k-1)\frac{\alpha\varepsilon^2}{2}\pi}) \geq \varepsilon\bar{L}(e^{k\frac{\alpha\varepsilon^2}{2}\pi} - 1)$$

for some $\bar{L} > 0$. Then $w = w_1 = \varepsilon K$ will be reached for some k satisfying

$$k \geq \frac{2}{\alpha\pi\varepsilon^2}\log\left(\frac{K}{\bar{L}} + 1\right). \tag{3.4}$$

Now, let us count the time for the synthesis: for each $\Delta\theta = \pi$, since

$$\frac{d\theta}{dt} = \frac{1-\varepsilon^2\beta}{\varepsilon} \geq \frac{1}{\varepsilon} - \varepsilon K_2,$$

we have

$$\Delta\theta = \pi \geq \Delta T \left(\frac{1}{\varepsilon} - \varepsilon K_2\right).$$

Here, ΔT is the time needed for $\Delta\theta = \pi$. Hence, $\Delta T \leq \dfrac{\pi\varepsilon}{1-\varepsilon^2 K_2}$.

At each step (2b), (2c) of the synthesis defined in Sec. 2.1.3, we need a time

$$\Delta T + 2\varepsilon \leq \frac{\pi\varepsilon}{1-\varepsilon^2 K_2} + 2\varepsilon \leq M\varepsilon$$

for some $M > 0$. Hence the total time τ to go to w_1 with the synthesis satisfies

$$\tau \leq kM\varepsilon \leq \frac{2M}{\alpha\pi\varepsilon}\log\left(\frac{K}{\overline{L}} + 1\right) + M\varepsilon,$$

which proves the result.

4. Normal forms for $n \geq 4$ and the proof of Theorem 4

In this section, our aim is to prove constructively Theorem 4. For this, we will generalize the "normal coordinates" and normal forms of the papers [2, 3]. Once this is done, the constructive proof of Theorem 4 is exactly the same as the proof of Theorem 1 in [5]. We will not recall it.

4.1. **Smooth diagonalization of one-parameter families of skew symmetric matrices.** In the sequel, the integer part of a real x is denoted by $[x]$, and A' means the transpose of the matrix A.

Any C^ω-real one-parameter family of normal (in particular, symmetric or skew symmetric) matrices can be analytically diagonalized. This is a classical fact (see [10]). Here, we will prove a C^∞ version of this fact, which is crucial in order to obtain the normal form of Theorem 7 below, and, in particular, to construct asymptotic optimal syntheses.

Let $\mathcal{S}$ denote the set of real smooth $n \times n$ one-parameter families of skew symmetric matrices,

$$A : t \to A(t), \quad A : [0,1] \to so(n).$$

Fix $t_0 \in [0,1]$, and let $j^k_{t_0}A$ be the k jet at t_0 of $A(\cdot)$. We denote the polynomial

$$j^k_{t_0}A(\tau) = A(t_0) + \tau\frac{dA}{dt}(t_0) + \cdots + \frac{\tau^k}{k!}\frac{d^kA}{dt^k}(t_0)$$

by the same symbol. By [10], $j^{k-1}_{t_0}A(\tau)$ has eigenvalues

$$\pm\sqrt{-1}\alpha_1^{k-1}(\tau),\ \ldots,\ \pm\sqrt{-1}\alpha_{[\frac{n}{2}]}^{k-1}(\tau)$$

(if n is odd, then 0 is also an additional eigenvalue) depending analytically on τ. Let us assume that, for some k, the $(k-1)$-jets at $\tau = 0$, $j^{k-1}\alpha_1^{k-1}, \ldots, j^{k-1}\alpha_{[\frac{n}{2}]}^{k-1}$ satisfy:

- if $i \neq j$, then $j^{k-1}\alpha_i^{k-1} \neq \pm j^{k-1}\alpha_j^{k-1}$ (n is even or odd),
- moreover, $j^{k-1}\alpha_i^{k-1} \neq 0$ (the zero jet) for all i (n is odd).

Then, we say that $A(\cdot)$ is regular at t_0. If this is true for all $t_0 \in [0,1]$, then $A(\cdot)$ is said to be *regular*.

Theorem 6. *The C^∞-families $A \in \mathcal{S}$ that are regular can be smoothly block-diagonalized* (*by conjugation with a smooth one-parameter family of orthogonal matrices*).

We will give the proof in the case of real skew symmetric matrices only, but the proof in the general case of (complex-valued) one-parameter families of Hermitian matrices is similar.

In this section, $\mathcal{S}_0$ will denote the space of germs at the origin ($t = 0$), of one-real-parameter families of skew symmetric matrices $A(t)$, and $j^k A(t)$ will be the k-jet at 0 of $A(t)$. $\mathcal{C}_t$ will denote the ring of germs at the origin of smooth functions of t, and $\mathcal{M}$ will be the maximal ideal of $\mathcal{C}_t$.

We denote by J the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. For $\alpha = (\alpha_1, \ldots, \alpha_{[\frac{n}{2}]})$, we denote by $\mathcal{D}_\alpha$ or $\mathcal{D}(\alpha)$ the block-diagonal matrix:

$$\mathcal{D}_\alpha = \text{Block-diag}(\alpha_1 J, \ldots \alpha_{\frac{n}{2}} J) \qquad \text{if } n \text{ is even},$$
$$\mathcal{D}_\alpha = \text{Block-diag}(\alpha_1 J, \ldots \alpha_{[\frac{n}{2}]} J, 0) \qquad \text{if } n \text{ is odd}.$$

Then $\mathcal{D}_{\alpha(\cdot)}$ will denote typical elements of $\mathcal{S}_0$ that are block-diagonal. Let such $\mathcal{D}_{\alpha(\cdot)}$ be fixed, so that the collection of elements $\alpha_i \pm \alpha_j$, $i, j = 1, \ldots, [\frac{n}{2}]$, $i \neq j$ *have nonzero* $(k-1)$*-jets at the origin* (if n is odd, we need also that the α_i's have nonzero $(k-1)$-jets at the origin). Then we have the following two lemmas.

Lemma 1. *For all $W(t) \in \mathcal{M}^k\mathcal{S}_0$, there is a smooth family of skew symmetric matrices $K(t) \in \mathcal{M}\mathcal{S}_0$, and a block-diagonal family $\mathcal{D}_{\beta(\cdot)}$, where $\beta_i(\cdot) \in \mathcal{M}^k$, such that*

$$W = K' D_\alpha + D_\alpha K + D_\beta. \tag{4.1}$$

Proof. We treat only the case of even n (the case of odd n is similar). The matrix K is represented by a $(\frac{n}{2} \times \frac{n}{2})$-matrix $\hat{K}$, whose entries are (2×2)-matrices $\hat{K}_{i,j} = \left(\begin{smallmatrix} a_{i,j} & b_{i,j} \\ c_{i,j} & d_{i,j} \end{smallmatrix}\right)$. Let us set $T = K' D_\alpha + D_\alpha K$. Then it is easy to verify that $\hat{T}_{i,i} = 0$, and for $i \neq j$,

$$\hat{T}_{i,j} = \alpha_i J \hat{K}_{i,j} - \alpha_j \hat{K}_{i,j} J.$$

This shows first that D_β is a block diagonal matrix with the entries $\hat{W}_{i,i}$ that have their own entries in $\mathcal{M}^k$ by the assumption. For the nondiagonal blocks, Eq. (4.1) becomes

$$\hat{W}_{i,j} = \alpha_i J \hat{K}_{i,j} - \alpha_j \hat{K}_{i,j} J.$$

This is a linear equation in $a_{i,j}$, $b_{i,j}$, $c_{i,j}$, and $d_{i,j}$. The determinant of the 4×4 matrix B of this linear equation is easily computed: it equals

$$\delta = (\alpha_i - \alpha_j)^2(\alpha_i + \alpha_j)^2.$$

The inverse B^{-1} has entries that are real multiples of the functions

$$a_1 = \frac{\alpha_i}{(\alpha_i)^2 - (\alpha_j)^2}, \quad b_1 = \frac{\alpha_j}{(\alpha_i)^2 - (\alpha_j)^2}.$$

But $a_1 t^k$ and $b_1 t^k$ both belong to $\mathcal{C}_t$. Let us prove this for $a_1 t^k$ only:

$$a_1 t^k = \frac{t^k}{2}\left(\frac{1}{\alpha_i - \alpha_j} + \frac{1}{\alpha_i + \alpha_j}\right).$$

The result follows from the assumption that $\alpha_i - \alpha_j$ and $\alpha_i + \alpha_j$ do not have a zero $(k-1)$-jet and hence are not in $\mathcal{M}^k$. This implies also that $K \in \mathcal{M}\mathcal{S}_0$. □

Lemma 2. *Let $M = D(\alpha) + \tilde{M}$ with $\tilde{M} \in \mathcal{M}^k\mathcal{S}_0$. Then, there is a smooth one-parameter family of orthogonal matrices $G(t)$, with $G(0) = \mathrm{Id}$, and a smooth one-parameter family of block-diagonal matrices $D(\bar{\alpha}(t))$, with $j^{k-1}\alpha_i(t) = j^{k-1}\bar{\alpha}_i(t)$ for all $i = 1, \ldots, [\frac{n}{2}]$, such that*

$$G' M G = D(\bar{\alpha}).$$

Proof. The proof is based upon the homotopy method (standard in real C^∞ singularity theory).

Let by M_τ, $\tau \in [0,1]$, $M_\tau = D(\alpha(t)) + \tau\tilde{M}(t)$ join $M_0 = D(\alpha(t))$ and $M_1 = D(\alpha(t)) + \tilde{M}(t)$, and look for a homotopy of orthogonal matrices $G_\tau(t)$ and for block-diagonal matrices $D_\tau = D(\alpha_\tau(t))$ such that $\alpha_\tau(t) = \alpha(t) + \tau\hat{\alpha}(t,\tau)$ and

$$M_\tau(t) = G'_\tau(t) D(\alpha_\tau(t)) G_\tau(t). \tag{4.2}$$

Differentiating (4.2) w.r.t. τ gives:

$$\tilde{M} = \frac{dG'_\tau}{d\tau} D_\tau G_\tau + G'_\tau D_\tau \frac{dG_\tau}{d\tau} + G'_\tau \frac{dD_\tau}{d\tau} G_\tau.$$

Conjugating this equality by $G_\tau^{-1} = G'_\tau$, we obtain

$$G_\tau \tilde{M} G'_\tau = K'_\tau D_\tau + D_\tau K_\tau + \frac{dD_\tau}{d\tau}, \tag{4.3}$$

where $K_\tau = \dfrac{dG_\tau}{d\tau} G_\tau^{-1} \in so(n)$ and the left-hand side belongs to $\mathcal{M}^k\mathcal{S}_0$ for all τ.

We consider the constant one-parameter family $\bar{G}(t) = g$ (depending on the parameter $g \in SO(n)$), and the family of diagonal matrices $\Delta_b = D(\alpha(t) + t^{k-1}b)$ with $b \in \mathbb{R}^{[\frac{n}{2}]}$.

Given such $\bar{G}$, Δ_b, we consider the following equation in $\bar{K}$, $\bar{D}$ (similar to (4.3) and (4.1)):

$$\bar{G}\tilde{M}\bar{G}' = \bar{K}'(t, g, b)\Delta_b + \Delta_b\bar{K}(t, g, b) + \bar{D}(t, g, b).$$

According to Lemma 1, this equation has a solution, defined in a neighborhood of the origin in the (t, b) space and for all $g \in SO(n)$. From the proof of Lemma 1, this solution $\bar{K}$, $\bar{D}$ is smooth with respect to all the arguments t, g, and b.

Then the differential equations

$$\frac{dg}{d\tau}g^{-1} = \bar{K}(g, t, b), \quad t^{k-1}\frac{dD(b)}{d\tau} = \bar{D}(g, t, b)$$

provide a smooth vertical vector field V on the fibers of the product bundle:

$$\pi : SO(n) \times \mathbb{R}^{[\frac{n}{2}]} \times \mathbb{R} \to \mathbb{R}, \quad (g, b, t) \mapsto t.$$

This is true since the entries of the block-diagonal matrix $\bar{D}(g, t, b)$ belong to $\mathcal{M}^k$ for fixed g and b.

At $t = 0$, the vector field V vanishes (this follows from the proof of Lemma 1 for $\bar{K}$, and from the fact that the entries of $\bar{D}$ belong to $\mathcal{M}^k$). Therefore, the flow of V with initial conditions $g(t, 0) = \mathrm{Id}$, $b(t, 0) = 0$ is (for t close to zero) well defined up to $\tau = 1$, and it defines a one-parameter family G_τ, D_τ satisfying (4.3). The fact that $j^{k-1}\alpha_i(t) = j^{k-1}\bar{\alpha}_i(t)$ is a consequence of the fact that $b(0, \tau) = 0$ for all τ, since the entries of $\bar{D} \in \mathcal{M}^k$. The lemma is proved. □

Proof of Theorem 6. We leave the details of the case of odd n to the reader. It is easy to show that Lemmas 1 and 2 already imply that if a family $A \in \mathcal{S}$ is regular, then for all $t_0 \in [0, 1]$, it can be smoothly block-diagonalized in a neighborhood of t_0. Then it remains only to construct a global block-diagonalization. For this, it is sufficient to glue two diagonalizations defined on a pair of overlapping intervals. If $A(t)$ is regular at t_0, then the points θ where not only k-jets but eigenvalues themselves are distinct, are dense near t_0. Take such θ in the intersection of the overlapping intervals.

Changing a diagonalization $U_1(t)$ on one of the intervals if necessary, we may assume that $U_1(\theta)^{-1}U_2(\theta)$ lies in the connected component of the identity of the isotropy subgroup $St(\theta) \subset O(n)$ of the diagonal matrix $U_1(\theta)^{-1}A(\theta)U_1(\theta)$.

Using a smooth homotopy $V(t) \in St(t)$ defined in a neighborhood $N(\theta)$ of θ, such that $V(t) = \mathrm{Id}$ for $t < \theta - \varepsilon$ and $V(t) = U_2(t)U_1(t)^{-1}$ for $t > \theta + \varepsilon$ (ε is small), the family:

$$\begin{array}{ll} U_1(t), & t < \theta - \varepsilon, \\ V(t)U_1(t), & \theta - \varepsilon \le t \le \theta + \varepsilon, \\ U_2(t), & t > \theta + \varepsilon, \end{array}$$

is a smooth block-diagonalization on the union of the two intervals. □

4.2. Normal coordinates and normal forms. Proof of Theorem 4.

Proposition 2. *Let Γ be fixed and $(\Delta, g) \in GS^\infty$ be as defined in Sec.* 2. *Then there exists a global germ of coordinate system (ξ, w) along Γ such that the following assertions hold*:

1. $\Gamma(t) = (0, t)$, $\Delta(\Gamma(t)) = \operatorname{Ker} dw$, $g(\Gamma(t)) = \sum\limits_{i=1}^{n-1} d\xi_i^2$;
2. *geodesics satisfying the Pontryagin maximum principle transversality conditions w.r.t. Γ are straight lines through Γ contained in the horizontal planes $S_{w_0} = \{w = w_0\}$*;
3. *for sufficiently small ε, the sub-Riemannian cylinders*
$$C_\varepsilon = \{q \mid d(q, \Gamma) \leq \varepsilon\}$$
(*where d is the sub-Riemannian distance*) *are the Riemannian cylinders* $\left\{\sum\limits_{i=1}^{n-1} \xi_i^2 \leq \varepsilon^2\right\}$.

Such a coordinate system is called a pre-normal coordinate system.

This proposition is easy and has already been stated several times (see [1–3, 5]).

Prenormal coordinate systems are unique up to coordinate changes of the form
$$(\tilde{\xi}, w) = (U(w)\xi, w),$$
where $U(w)$ is a smooth curve in $SO(n-1)$.

Now let $A(t)$ be the field of linear endomorphisms of $\Delta(\Gamma(t))$ associated with $\Sigma = (\Gamma, \Delta, g)$. If a prenormal coordinate system (ξ, w) is given, let us denote by $\bar{A}(t)$ the matrix of $A(t)$ with respect to the orthonormal frame $\left(\dfrac{\partial}{\partial \xi_1}, \dots, \dfrac{\partial}{\partial \xi_{n-1}}\right)$.

Proposition 3 (normal coordinates). *Assume that the one-parameter family of skew symmetric matrices $\bar{A}(t)$ is regular* (*in the sense of Sec.* 4.1). *Then, there is a global* (*germ of*) *coordinate system (ξ, w) along Γ* (*called a normal coordinate system*) *such that*:

1. (ξ, w) *is prenormal*;
2. *the matrices $\bar{A}(t)$ are skew-symmetric and block-diagonal.*

Proof. We start in a given pre-normal coordinate system (ξ, w). By Theorem 6, we can block diagonalize $\bar{A}(t)$ by a smooth orthogonal transformation $U(t)$.

Setting $(\tilde{\xi}, w) = (U(w)\xi, w)$ does the job, since these coordinate changes do not affect the properties (1)–(3) of prenormal coordinates. □

Normal coordinates are unique up to coordinate changes of the form $(\tilde{\xi}, w) = (T(w)\xi, w)$, where $T(w)$ is a smooth curve in the natural maximal torus of $SO(n-1)$.

In a normal coordinate system, the coordinates (ξ, w) split into coordinates (x, y, w) (n is odd), or (x, y, z, w) (n is even) such that $\xi = (x, y) = (x_1, y_1, \dots, x_{\frac{n-1}{2}}, y_{\frac{n-1}{2}})$ (or $\xi = (x, y, z)$, $\xi = (x_1, y_1, \dots, x_{[\frac{n-1}{2}]}, y_{[\frac{n-1}{2}]}, z)$, $z \in \mathbb{R}$). The two-dimensional real eigenspaces of $\bar{A}(t)$ are span $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right\}$, and the unavoidable kernel (for even n) is span $\left\{ \frac{\partial}{\partial z} \right\}$.

If normal coordinates are given, and an orthonormal frame field $\mathcal{F} = (F_1, \dots, F_{n-1})$ is also given, then, in coordinates, we write the frame

$$\mathcal{F} = (F_1, \dots, F_{n-1}) = F\frac{\partial}{\partial \xi} + L\frac{\partial}{\partial w},$$

which means that

$$F_j = \sum Q_{i,j}\frac{\partial}{\partial \xi_i} + L_j\frac{\partial}{\partial w}.$$

Then the matrices $Q(\xi, w)$ and $L(\xi, w)$ have k-jets with respect to ξ:

$$Q(\xi, w) = Q_0(\xi, w) + Q_1(\xi, w) + Q_2(\xi, w) + \dots,$$
$$L(\xi, w) = L_0(\xi, w) + L_1(\xi, w) + L_2(\xi, w) + \dots,$$

where $Q_i(\xi, w)$ and $L_i(\xi, w)$ are homogeneous polynomials of degree i in the variables x_j, y_j (and z if n is even).

Also, let us again denote by $\pm\sqrt{-1}\alpha_1(w), \dots, \pm\sqrt{-1}\alpha_{[\frac{n-1}{2}]}(w)$ (and 0) the eigenvalues of $\bar{A}(w)$.

Theorem 7 (normal form)**.** *Under the same assumption that $\bar{A}(\cdot)$ is a regular family, if (x, y, w) (respectively, (x, y, z, w)) is a fixed normal coordinate system, then there is a unique orthonormal frame field*

$$\mathcal{F} = (F_1, \dots, F_{n-1}) = Q\frac{\partial}{\partial \xi} + L\frac{\partial}{\partial w}$$

for the sub-Riemannian metric, with the following properties:

1. *Q is symmetric*;
2. $Q_0(\xi, w) = \mathrm{Id}$;
3. $Q(\xi, w) \cdot \xi = \xi$;
4. $L(\xi, w) \cdot \xi = 0$;
5. $Q_1 = 0$;
6. $L_0 = 0$;
7. *if n is odd, then*

$$L_1 = \left(\frac{\alpha_1(w)}{2} y_1,\ -\frac{\alpha_1(w)}{2} x_1,\ \dots,\ \frac{\alpha_{\frac{n-1}{2}}(w)}{2} y_{\frac{n-1}{2}},\ -\frac{\alpha_{\frac{n-1}{2}}(w)}{2} x_{\frac{n-1}{2}} \right);$$

if n *is even, then*

$$L_1 = \left(\frac{\alpha_1(w)}{2} y_1, -\frac{\alpha_1(w)}{2} x_1, \dots, \frac{\alpha_{[\frac{n-1}{2}]}(w)}{2} y_{[\frac{n-1}{2}]}, -\frac{\alpha_{[\frac{n-1}{2}]}(w)}{2} x_{[\frac{n-1}{2}]}, 0 \right).$$

In fact, this is a generalization of the 3-dimensional normal form (2.2): actually, (2.2) can be easily obtained from Theorem 7.

The proof of Theorem 7 is exactly the same as in [5].

Moreover, based upon this theorem, we have the following proposition, which is proved (constructively, i.e., exhibiting an asymptotic optimal synthesis) exactly as Theorem 1 in [5].

Proposition 4. *Assume that* $\bar{A}(\cdot)$ *is a regular family, and assume that the fundamental* 2*-form* $\alpha(t) \neq 0$ *for all* $t \in [0,1]$ assumption (c)*. Then asymptotics* (1.1) *holds for the metric complexity, and the leading coefficient of* $MC_\Sigma(\varepsilon, T)$ *is a* C^1 *smooth function of* T*, piecewise* C^∞*.*

Lemma 3. *For all* $t_0 \in [0,1]$*, the mapping*

$$J^{k+1}_{t_0} GS^\infty \to J^k \mathcal{S}_0, \quad j^{k+1}_{t_0}(\Delta, g) \to j^k_{t_0} \bar{A},$$

is a submersion.

Proof. Take coordinates (ξ, w) along Γ such that $\Gamma(t) = (0,t)$, $\Delta(\Gamma(t)) = \operatorname{Ker} dw$, and $g|_{\Gamma(t)} = \sum\limits_{i=1}^{n-1} d\xi_i^2$. Then the forms ω (defined in Sec. 1.3) can be written as follows:

$$\omega = dw + \sum_{i=1}^{n-1} f_i(\xi, w) d\xi_i.$$

The matrix $\bar{A}(w)$ of $\alpha = d\omega_{|\Gamma(t)}$ in these coordinates is

$$\bar{A}(w)_{i,j} = \left(\frac{\partial f_i}{\partial \xi_j} - \frac{\partial f_j}{\partial \xi_i} \right)(0, w).$$

The lemma is proved. □

Therefore, Theorem 4 follows from Proposition 4 and the following lemma, the proof of which is easy and left to the reader.

Lemma 4. *The elements of* $J^{k-1}\mathcal{S}_0$*, i.e., the* $(k-1)$*-jets* $j_0^{k-1}\bar{A}$ *at* $t = 0$ *of one-parameter families* $\bar{A}(t)$ *that are not regular at* $t = 0$ *form an algebraic subset of* $J^{k-1}\mathcal{S}_0$ *of codimension* $3k$ *in* $J^{k-1}\mathcal{S}_0$*.*

References

1. A. A. Agrachev, H. E. A. Chakir, and J. P. Gauthier, Sub-Riemannian metrics on $\mathbb{R}^3$. In: *Geometric Control and Nonholonomic Mechanics*, Mexico City (1996), pp. 29–76, *Proc. Can. Math. Soc.* **25** (1998).

2. A. A. Agrachev and J. P. Gauthier, Sub-Riemannian metrics and isoperimetric problems in the contact case. *J. Math. Sci.* **103** (2001), No. 6, pp. 639–663.
3. G. Charlot, Quasi-contact SR metrics: normal form in $\mathbb{R}^{2n}$. Wave front and caustic in $\mathbb{R}^4$. *Acta Appl. Math.* **74** (2002), No. 3, pp. 217–263.
4. H. E. A. Chakir, J. P. Gauthier, and I. A. K. Kupka, Small sub-Riemannian balls on $\mathbb{R}^3$. *J. Dynam. Control Systems* **2** (1996), No. 3, pp. 359–421.
5. J. P. Gauthier, F. Monroy-Perez, and C. Romero-Melendez, On complexity and motion planning for corank one SR metrics. *Preprint* (2003).
6. M. Gromov, Carnot–Caratheodory spaces seen from within, *Birkhäuser* (1996), pp. 79–323.
7. F. Jean, Complexity of nonholonomic motion planning. *Int. J. Control* **74** (2001), No. 8, pp. 776–782.
8. ———, Entropy and complexity of a path in SR geometry (in press).
9. F. Jean and E. Falbel, Measures and transverse paths in SR geometry (in press).
10. T. Kato, Perturbation theory for linear operators. *Springer-Verlag* (1966), pp. 120–122.
11. J. P. Laumond, S. Sekhavat, and F. Lamiraux, Robot motion planning and control. *Lect. Notes Control Inform. Sci.* **229**, *Springer-Verlag* (1998).

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