An Adaptive High-gain Observer for Nonlinear Systems

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Abstract

The main contribution of this paper is to provide a solution to the noise sensitivity of high-gain observers. We propose a nonlinear observer that possesses simultaneously the properties of 1) the extended Kalman filter, which behaves well with respect to noise, and 2) the high-gain extended Kalman filter that is performant with respect to large perturbations.

The idea is to adapt the gain in terms of the innovation.

We prove a general convergence result, propose guidelines to practical implementation and show simulation results for an example.

Key words: Nonlinear observer; Adaptive high-gain observer; Kalman filtering.

1 Introduction

We deal with an observer for nonlinear systems. This question is usually addressed either in a stochastic or deterministic setting. The stochastic representation (see [6, 20, 21] for rigorous definitions):

$$\begin{cases} dX(t) \;=\; f(X(t), u(t))dt + Q^{\frac{1}{2}} dW(t) \\ dY(t) \;=\; h(X(t), u(t)) + R^{\frac{1}{2}} dV(t) \end{cases} \tag{1}$$

naturally leads us to consider the extended Kalman filter (EKF) algorithm because of its noise filtering properties (see e.g. [21]). Let us consider the deterministic representation of (1):

$$\begin{cases} \frac{x(t)}{dt} \;=\; f(x(t), u(t)) \\ y(t) \;=\; h(x(t), u(t)). \end{cases} \tag{2}$$

The analytical study of the EKF shows that convergence of the estimated state to the real state is theoretically guaranteed provided that the estimate lies into the neighborhood of the real trajectory (see [4, 11, 22]). In other words, the convergence of the observer is theoretically justified only if we already have a somewhat precise of what the state is, and that no unpredicted state jumps occur.

On the other hand, the high-gain formalism allows us to build globally convergent observers: the initial estimate can be chosen anywhere in a compact subset of the state space. This approach essentially requires two ingredients (see [2, 10, 16–18, 23]):

- the system under consideration must have a "strong observability property". This property is generic when the number of outputs is greater than the number of inputs (see [18]). On the contrary, if there are less inputs than outputs, this property is very restrictive. But in both cases, strongly observable systems may be put under the observability canonical form used in this paper,
- the observer algorithm is embedded with the *high-gain structure*, based on a fixed scalar parameter, called the high-gain parameter denoted by θ (Cf. Subsection 2.3).

In this formalism, as shown in [18], convergence takes place whenever θ is set at a large enough value. This algorithm though has an important drawback: a high valued θ leads to noise amplification.

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Our purpose is to combine the noise filtering properties of the extended Kalman filter with the global convergence properties of the high-gain extended Kalman filter. The high-gain structure should be used only when necessary. In order to achieve this goal, we let θ vary with time. We adapt it under the guise of the differential equation:

$$\frac{d\theta(t)}{dt} = \mathcal{F}(\theta, \mathcal{I}).$$

When $\theta = 1$ the high-gain EKF is a classic EKF. Thus θ must evolve between 1 and a high enough value, whose existence has to be proven.

High-gain observers based on such ideas were proposed in [1, 2, 10]. They are of the Luenberger style: the correction gain is computed offline. In Bulliger and Allgöwer [10], the problem of the tuning of θ is addressed. The parameter increases until convergence becomes effective. In this algorithm, θ cannot go down, the efficiency with respect to noise is not the main purpose there. In Praly *et al.* [2] the adaptation is driven by the evolution of the model nonlinearities. They model the way the Lipschitz parameter of $b(x, u)$ changes (refer to Subsection 2.2), and adapt θ accordingly. In Khalil *et al.* [1], the efficiency with respect to noise is the objective. The difference lies in the type of observer used. The local convergence of the EKF allows us to define an observer that filters noise very efficiently.
Another method to attack the problem of the non-globally guaranteed convergence of the EKF was studied in Grizzle *et al.* [14]. They used an EKF together with numerical differentiation observers.

The type of system under consideration, and the observer are introduced in Section 2. The main theorem appears in Section 3 and the proof in Section 4. Finally, in Section 5, an illustrative example is used to give an account of the observer performances. Guidelines to the choice of parameters are provided.

2 Systems under consideration and observer definition

2.1 Notations

For a vector v, $diag(v)$ is a diagonal matrix, diagonal being equal to v.
The contribution of the variable u to the vector fields, and time dependencies are sometimes omitted to ease the reading of equations.

2.2 The observability canonical form

We suppose that the system is under the multiple inputs, single output observability form (3). As explained in [12, 17, 18], this form reflects the observability characteristics of the system (see also Remark 1). It is therefore pertinent in many situations since we want to apply observers to observable systems.
This structure is also a requirement in any proof of exponential convergence of high-gain observers: all the systems used in [1, 2, 10, 18, 23] have a structure similar to the one in system (3).

Single output systems are considered only for the clarity of the exposition. This observer construction, and the proof, remain valid in the multiple output case. Since there is no unique observability form in the multiple output case, the observer has to be adapted to each situation. Details on this topic can be found in [5]. The system under consideration is:

$$\begin{cases} \frac{dx}{dt} = A(u)\,x + b(x, u) \\ y = C(u)\,x \end{cases} \tag{3}$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$, $\mathcal{X}$ compact, $y(t) \in \mathbb{R}$ and $u(t) \in \mathcal{U}_{\text{adm}} \subset \mathbb{R}^{n_u}$ is a bounded function.
The matrices $A(u)$ and $C(u)$ are defined by:

$$A(u) = \begin{pmatrix} 0 & a_2(u) & 0 & \cdots & 0 \\ & 0 & a_3(u) & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & 0 & a_n(u) \\ 0 & & \cdots & & 0 \end{pmatrix}$$

$$C(u) = (a_1(u), 0, \cdots, 0)$$

with $0 < a_m \leq a_i(u) \leq a_M$ for any u in $\mathcal{U}_{\text{adm}}$. The vector field $b(x, u)$ is assumed to be compactly supported and to have the following triangular structure:

$$b(x, u) = \begin{pmatrix} b_1(x_1, u) \\ b_2(x_1, x_2, u) \\ \vdots \\ b_n(x_1, \ldots, x_n, u) \end{pmatrix}$$

We denote L_b the bound on the Jacobian matrix $b^*(x, u)$ of $b(x, u)$ (i.e. $\|b^*(x, u)\| \leq L_b$). Since $b(x, u)$ is compactly supported and u is bounded, b is Lipschitz w.r.t. x, uniformly in u: $\|b(x_1, u) - b(x_2, u)\| \leq L_b \|x_1 - x_2\|$.

Remark 1

- *When the matrix $A(u)$ is input driven, Luenberger style observers*[1] *cannot be used anymore. Kalman-like observers can be applied to such systems.*
- *The relevance of the form (3) is discussed in Section* 2 *of [11]. For instance, the Lipschitz assumption on b is not a restriction. This second point has to be developed:*

[1] i.e. having a fixed, correction gain computed offline.

(1) First in general, for physical reasons, the state space is bounded,
(2) As explained in [18], convergence of nonlinear observers on a non-compact space is a nonsense,
(3) Even if the state space is not bounded, the Lipschitz assumption is not required for the dynamic output stabilization.

2.3 Observer definition

The *extended Kalman filter with adaptive high-gain* is given by the system:

$$\begin{cases} \frac{dz}{dt} &= A(u)z + b(z,u) - S^{-1}C'R_\theta^{-1}(Cz - y(t)) \\ \frac{dS}{dt} &= -(A(u)+b^*(z,u))'S - S(A(u)+b^*(z,u)) \\ & \qquad +C'R_\theta^{-1}C - SQ_\theta S \\ \frac{d\theta}{dt} &= \mathcal{F}(\theta(t), \mathcal{I}_d(t)) \\ &= \mu(\mathcal{I}_d) \times \mathcal{F}_0(\theta) + \lambda(1-\mu(\mathcal{I}_d)) \times (1-\theta) \end{cases} \tag{4}$$

where

- $z(0) \in \chi$, $\theta(0) = 1$, and $\theta(t) \geq 1$,
- $S(0)$ is a $(n \times n)$ symmetric positive definite matrix, the second equation is therefore a Riccati equation,
- $b^*(z,u)$ denotes the Jacobian of the vector field $b(x,u)$ w.r.t. x, computed along the estimated trajectory.
- Q_θ and R_θ are defined as follows:
  let Q be a $(n \times n)$ symmetric positive definite matrix, $R > 0$ a scalar, and
  $$\Delta = diag\left(\left\{1, \frac{1}{\theta(t)}, \frac{1}{\theta(t)^2}, \dots, \frac{1}{\theta(t)^{n-1}}\right\}\right).$$
  Then $Q_\theta = \theta(t)\Delta^{-1}Q\Delta^{-1}$, and $R_\theta = \theta(t)^{-1}R$.
- The function $\mathcal{I}_d$ is called the innovation. It is the quantity:
  $$\mathcal{I}_d(t) = \int_{t-d}^{t} \|y(\tau) - \hat{y}(\tau)\|^2 d\tau$$
  · y is the actual output of the system (3), i.e. the measurements, and
  · $\hat{y}$ is a prediction of the output trajectory of the system (3), computed over the interval $[t-d, t]$ with initial state $z(t-d)$.
- The functions $\mathcal{F}_0$ and μ are:
  $$\mathcal{F}_0(\theta) = \begin{cases} \frac{1}{\Delta T}\theta^2 & if \ \theta \leq \theta_1 \\ \frac{1}{\Delta T}(\theta - 2\theta_1)^2 & if \ \theta > \theta_1 \end{cases} \tag{5}$$
  and, with $0 < \gamma_0 < \gamma_1$:
  $$\mu(\mathcal{I}_d) = \begin{cases} 0 & if \quad \mathcal{I}_d \leq \gamma_0 \\ \in [0;1] & if \ \ \gamma_0 < \mathcal{I}_d < \gamma_1 \\ 1 & if \quad \mathcal{I}_d \geq \gamma_1 \end{cases} \tag{6}$$
  $\mathcal{F}$ is such that,
  · when $\mathcal{I}_d(t) \geq \gamma_1$: θ increases towards $2\theta_1$ and is above θ_1 in a time less than ΔT, for any θ_1, and
  · when $\mathcal{I}_d(t) \leq \gamma_0$: θ decreases toward 1, at a rate set via the parameter λ.
  The function μ controls which part of $\mathcal{F}$ is active at a given time.

Remark 2 • *When we use (1) to represent the system, Q and R are the state and output noise covariance matrices, respectively. Hence, those two matrices are not meaningless parameters. They determine the noise filtering properties of the extended Kalman filter mode of the above defined observer.*
• *The definition of $\mathcal{F}$ looks cumbersome at first glance. It is, in fact, a simple function that meets our needs regarding the increase and the decrease of θ, and respects the requirements appearing in Subsection 4.2.*

3 The main theorem

3.1 Innovation

Our definition of innovation, $\mathcal{I}_d(t)$, is different from previous definitions (e.g. [9,15]). It is a quality measurement of the estimation error justified by Lemma 4.

Remark 3 *As is well known from the linear case, the matrix S is closely related to innovation, since the Gramm of observability of the linearized system is a lower bound of S. Therefore, one can suppose that S should be used instead of innovation. Unfortunately, in our result, we need to use the nonlinear innovation, according to the following Lemma.*

Lemma 4 *Let $x_1^0, x_2^0 \in \mathbb{R}^n$ and $u \in \mathcal{U}_{\mathrm{adm}}$. Let us consider the outputs $y\left(0, x_1^0, \cdot\right)$ and $y\left(0, x_2^0, \cdot\right)$ of system (3) with initial conditions respectively x_1^0 and x_2^0. Then the following property (called persistent observability) holds:*

$$\forall d > 0, \exists \lambda_d^0 > 0 \textit{ such that } \forall u \in L_b^1(\mathcal{U}_{adm})$$

$$\|\overset{0}{x_1} - \overset{0}{x_2}\|^2 \leq \frac{1}{\lambda_d^0}\int_0^d \|y\left(0, x_1^0, \tau\right) - y\left(0, x_2^0, \tau\right)\|^2 d\tau \tag{7}$$

Proof. A proof of this lemma in the continuous-discrete case can be found in [7]. ■

Let us set $x_1^0 = z(t-d)$, and $x_2^0 = x(t-d)$ then, with the notations of Subsection 2.3, Lemma 4 gives:

$$\|z(t-d) - x(t-d)\|^2 \leq \frac{1}{\lambda_d^0}\int_{t-d}^{t} \|y(\tau) - \hat{y}(\tau)\|^2 d\tau,$$

or, equivalently,

$$\|z(t-d) - x(t-d)\|^2 \leq \frac{1}{\lambda_d^0}\mathcal{I}_d(t).$$

That is to say, up to a multiplication by a constant, innovation at time t upper bounds the estimation error at time $t-d$.

3.2 Main result

Theorem 5 *For any time $T^*>0$ and any $\varepsilon^*>0$, there exists $0<d<T^*$, $\Delta T>0$, $\lambda>0$, and $\theta_1>0$ such that $\mathcal{F}$ is properly defined, and:*

for all $t \geq T^*$, and any couple $(x(0), z(0)) \in \chi \times \chi$:

$$\|x(t) - z(t)\|^2 \leq \varepsilon^* \mathrm{e}^{-a\,(t-T^*)} \tag{8}$$

where $a>0$ is a constant (independent from ε^*).

In order to prove this theorem, we study the Lyapunov function $\left(\varepsilon' S \varepsilon\right)(t)$. A change of variables is done to make θ more tractable. In this new system of coordinates, inequalities that represent the local and the global convergence properties are obtained. We then prove that the function $\mathcal{F}$ defined in (4) allows us to pass from one configuration to the other. More importantly, we show that positive values of θ_1 exists such that the observer converges globally. As a consequence, the algorithm is consistent. We reverse the change of variables to get inequality (8).

The overall proof is divided into two parts. We compute preliminary inequalities in Subsection 4.1. The articulation of the proof is explained in Subsection 4.2.

4 Proof of the theorem

4.1 Part 1

Recall that $\theta(t) \geq 1$, for all $t \geq 0$.
Let us denote $\varepsilon = z - x$ and consider the change of variables $\tilde{x} = \Delta x$, $\tilde{\varepsilon} = \Delta \varepsilon$, $\tilde{z} = \Delta z$, $\tilde{S} = \Delta^{-1} S \Delta^{-1}$, $\tilde{b}(.) = \Delta b(\Delta^{-1}.)$ and $\tilde{b}^*(\cdot) = \Delta b^*\left(\Delta^{-1}\cdot\right)\Delta^{-1}$.
We have the relations:

- $\Delta A = \theta A \Delta$, $A\Delta^{-1} = \theta \Delta^{-1} A$,
- $C\Delta = C$ in the single output case,
- $\frac{d\Delta}{dt} = -\frac{\dot{\theta}}{\theta} N \Delta$ and $\frac{d\Delta^{-1}}{dt} = \frac{\dot{\theta}}{\theta} N \Delta^{-1}$, where N is the $(n \times n)$ matrix $diag\left(\{0, 1, 2, \ldots, n-1\}\right)$.

The error dynamics in the new coordinates are:

$$\begin{aligned}\frac{d\tilde{\varepsilon}}{dt} = \theta \Big[&-\frac{\mathcal{F}(\theta, \mathcal{I}_d)}{\theta^2} N \tilde{\varepsilon} + A \tilde{\varepsilon} - \tilde{S}^{-1} C' R^{-1} C \tilde{\varepsilon} \\ &+ \frac{1}{\theta}\left(\tilde{b}(\tilde{z}, u) - \tilde{b}(\tilde{x}, u)\right)\Big],\end{aligned}$$

and

$$\begin{aligned}\frac{d\tilde{S}}{dt} = \theta \Big[&\frac{\mathcal{F}(\theta, \mathcal{I}_d)}{\theta^2}\left(N\tilde{S} + \tilde{S}N\right) - \left(A'\tilde{S} + \tilde{S}A\right) - \tilde{S}Q\tilde{S} \\ &- \frac{1}{\theta}\tilde{S}\tilde{b}^*(\tilde{z}) - \frac{1}{\theta}\tilde{b}^*(\tilde{z})'\tilde{S} + C'R^{-1}C\Big].\end{aligned} \tag{9}$$

Let us now establish a crucial inequality concerning the Lyapunov function $\tilde{\varepsilon}' \tilde{S} \tilde{\varepsilon}$:

$$\begin{aligned}\frac{d\tilde{\epsilon}'\tilde{S}\tilde{\epsilon}}{dt} = \theta \Big[&-\tilde{\epsilon}' C' R^{-1} C \tilde{\epsilon} - \tilde{\epsilon}' \tilde{S} Q \tilde{S} \tilde{\epsilon} \\ &+ \frac{2}{\theta} \tilde{\epsilon}' \tilde{S}\left(\tilde{b}(\tilde{z}, u) - \tilde{b}(\tilde{x}, u) - \tilde{b}^*(\tilde{z}, u)\tilde{\epsilon}\right)\Big]\end{aligned} \tag{10}$$

with $Q \geq q_m\, Id$ and considering that $\tilde{\varepsilon}' C' R^{-1} C \tilde{\varepsilon} \geq 0$

$$\begin{aligned}\frac{d}{dt}\left(\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}\right) \leq &-\theta q_m \tilde{\varepsilon}' \tilde{S}^2 \tilde{\varepsilon} \\ &+ 2\tilde{\varepsilon}'\tilde{S}\left(\tilde{b}(\tilde{z}, u) - \tilde{b}(\tilde{x}, u) - \tilde{b}^*(\tilde{z}, u)\tilde{\varepsilon}\right).\end{aligned} \tag{11}$$

In order to continue our computations and actually find an upper bound for $\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}$ we need:

- extra information on the matrix $\tilde{S}$, and
- to upper bound the term $\left(\tilde{b}(\tilde{z}, u) - \tilde{b}(\tilde{x}, u) - \tilde{b}^*(\tilde{z}, u)\tilde{\varepsilon}\right)$.

This latter step can be performed in two different ways. One translates the local convergence, and the other expresses the global convergence (C.f. Subsection 4.2).

We want to study the properties of $\tilde{S}$ independently from θ. We remove it from equation (9) by means of the following timescale modification: $d\tau = \theta(t)\, dt$. The corresponding notation is $\overline{S}(\tau) = \widetilde{S}(t)$. A few computations gives us the Riccati equation in the τ time scale:

$$\begin{aligned}\frac{d\bar{S}}{d\tau} = &\frac{\mathcal{F}(\bar{\theta}, \bar{\mathcal{I}})}{\theta^2}\left(N\bar{S} + \bar{S}N\right) - \left(A'\bar{S} + \bar{S}A\right) + C'R^{-1}C \\ &- \bar{S}Q\bar{S} - \frac{1}{\bar{\theta}}\left(\bar{S}\tilde{b}^*(\bar{z}, \bar{u}) + \tilde{b}^*(\bar{z}, \bar{u})'\bar{S}\right).\end{aligned} \tag{12}$$

The information we need is given in the Lemma below.

Lemma 6 *([18]) Let us consider the Riccati equation (12) together with the assumptions:*

(1) the functions $a_i(u(t))$, $\left|\tilde{b}^*_{i,j}(\overline{z}, \overline{u})\right|$, $\left|\frac{\mathcal{F}(\overline{\theta}, \overline{\mathcal{I}})}{\overline{\theta}^2}\right|$ are smaller than $a_M > 0$,
(2) $a_i(u(t)) > a_m > 0$,
(3) $aId \leq S(0) \leq bId$,
(4) $\theta(0) = 1$.

Then two constants $0 < \alpha < \beta$ exists such that, the solution of equation (12) satisfies, for all $\tau > 0$, the inequality

$$\alpha Id \leq \overline{S}(\tau) \leq \beta Id. \tag{13}$$

Thus, this relation is true in the original time scale t .

Proof. The two bounds α and β are obtained as the minimum and maximum elements respectively out of a set of three.

(1) since $\theta(0) = 1$ and $S(0)$ is assumed bounded, so is $\bar{S}(0)$,
(2) for a given $\tau_0 > 0$, Lemma 6.2.18 of [18], (page 113), gives us bounds for $\bar{S}(\tau_0)$ for $\tau \geq \tau_0$. Assumptions (1) and (2) are necessary for this purpose , C.f. Lemma 6.2.14,
(3) bounds for $\tau \in]0; \tau_0[$ are obtained from the expressions of $\frac{d\bar{S}}{d\tau}$, $\frac{d\bar{S}^{-1}}{d\tau}$, and Gronwall's lemma.

■

It is well known that α and β can be expressed as some function of the Gramm observability matrix and the Gramm controllability matrix (see [18]).

4.2 Part 2

Let T^*, and ε^* be two positive scalars.
First of all let us choose a time horizon d (in $\mathcal{I}_d(t)$), and a time T such that $0 < d < T < T^*$. Set $\Delta T = T - d$ and choose $\lambda > 0$ in system (4). $\mathcal{F}$ is such that:

$$\left| \frac{\mathcal{F}(\theta, \mathcal{I}_d)}{\theta^2} \right| < \frac{1}{\Delta T} + \frac{\lambda}{4}.$$

Since this bound is independent from θ, we can use Lemma 6 to obtain α and β, independently from θ. Inequality (11), and $\tilde{S} \geq \alpha\, Id$ give:

$$\frac{d\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t)}{dt} \leq -\alpha q_m \theta \tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t) + 2\tilde{\varepsilon}'\tilde{S}\left(\tilde{b}(\tilde{z}) - \tilde{b}(\tilde{x}) - \tilde{b}^*(\tilde{z})\tilde{\varepsilon}\right). \tag{14}$$

From (14) we can deduce two inequalities: the first one, global, will be used mainly when $\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t)$ is not in the neighborhood of 0 and θ is large. The second one, local, will be used when $\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t)$ is small, whatever the value of θ. The Lipschitz [2] assumption of (3) gives:

$$\left\| \tilde{b}(\tilde{z}) - \tilde{b}(\tilde{x}) - \tilde{b}^*(\tilde{z})\tilde{\varepsilon} \right\| \leq 2L_b \|\tilde{\varepsilon}\|. \tag{15}$$

(15) and Lemma 6 turns (14) into the "global inequality":

$$\frac{d\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t)}{dt} \leq -\left(\alpha q_m \theta - 4\frac{\beta}{\alpha}L_b\right)\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t). \tag{16}$$

[2] The change of coordinates keeps the Lipschitz constant unchanged.

We now build the "local inequality". According to Lemma 5.2 of [11], (page 284):

$$\left\| \tilde{b}(\tilde{z}) - \tilde{b}(\tilde{x}) - \tilde{b}^*(\tilde{z})\tilde{\varepsilon} \right\| \leq K\theta^{n-1} \|\tilde{\varepsilon}\|^2,$$

for some $K > 0$. Since $1 \leq \theta \leq 2\theta_1$, (14) gives also:

$$\frac{d\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t)}{dt} \leq -\alpha q_m \tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t) + 2K(2\theta_1)^{n-1}\left\|\tilde{S}\right\| \|\tilde{\varepsilon}\|^3.$$

We notice that $\|\tilde{\varepsilon}\|^3 = \left(\|\tilde{\varepsilon}\|^2\right)^{\frac{3}{2}} \leq \left(\frac{1}{\alpha}\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t)\right)^{\frac{3}{2}}$, thus

$$\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t) \leq -\alpha q_m \tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t) + \frac{2K(2\theta_1)^{n-1}\beta}{\alpha^{\frac{3}{2}}}\left(\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t)\right)^{\frac{3}{2}}. \tag{17}$$

This is the "Local inequality". It is interpreted via Lemma 5.1 of [11]: if

$$\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(T_0) \leq \frac{\alpha^5 q_m^2}{16\, K^2 (2\theta_1)^{2n-2}\beta^2}$$

for some $T_0 > 0$, then for all $t \geq T_0$:

$$\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t) \leq 4\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(T_0)\, \mathrm{e}^{-\alpha q_m (t - T_0)}.$$

Consequently, if we find a $\gamma \in \mathbb{R}^{+*}$ such that

$$\gamma \leq \frac{1}{(2\theta_1)^{2n-2}} \min\left(\frac{\alpha\varepsilon^*}{4}, \frac{\alpha^5 q_m^2}{16\, K^2\beta^2}\right) \tag{18}$$

then $\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(T_0) \leq \gamma$ implies, for all $t \geq T_0$

$$\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t) \leq \frac{\alpha\varepsilon^*}{(2\theta_1)^{2n-2}}\mathrm{e}^{-\alpha q_m (t - T_0)}. \tag{19}$$

Now, from (16), since $\theta \geq 1$,

$$\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(T) \leq \tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(0)\, \mathrm{e}^{\left(-\alpha q_m + 4\frac{\beta}{\alpha}L_b\right)T}$$

and if we suppose $\theta \geq \theta_1$ for $t \in [T, T^*]$, $T^* > T$, using (16) again

$$\begin{aligned} \tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(T^*) &\leq \tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(0)\, \mathrm{e}^{\left(-\alpha q_m + 4\frac{\beta}{\alpha}L_b\right)T}\mathrm{e}^{\left(-\alpha q_m\theta_1 + 4\frac{\beta}{\alpha}L_b\right)(T^* - T)} \quad (20) \\ &\leq M_0 \mathrm{e}^{-\alpha q_m T}\mathrm{e}^{4\frac{\beta}{\alpha}L_b T^*}\mathrm{e}^{-\alpha q_m \theta_1 (T^* - T)} \end{aligned}$$

where $M_0 = \sup_{x,z \in \chi} \varepsilon' S \varepsilon(0) = \sup_{x,z \in \chi} \tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(0)$.
Let us choose θ_1 and γ to satisfy both

$$M_0 \mathrm{e}^{-\alpha q_m T}\mathrm{e}^{4\frac{\beta}{\alpha}L_b T^*}\mathrm{e}^{-\alpha q_m \theta_1 (T^* - T)} \leq \gamma \tag{21}$$

and (18). This is possible since $e^{-\text{cte}\times\theta_1} < \frac{\text{cte}}{\theta_1^{2n-2}}$ for θ_1 large enough. In the definition of $\mathcal{F}$, equation (4), we set $\Delta T = T - d$ and $\gamma_1 = \frac{\lambda_d^0 \gamma}{\beta}$.

We claim that $T_0 \leq T^*$ exists such that $\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(T_0) \leq \gamma$. Indeed, if $\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(T_0) > \gamma$ for all $T_0 \leq T^*$ then thanks to Lemma 4:

$$\begin{aligned}\gamma < \tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(T_0) &\leq \beta \left\|\tilde{\varepsilon}(T_0)\right\|^2 \\ &\leq \beta \left\|\varepsilon(T_0)\right\|^2 \leq \frac{\beta}{\lambda_d^0}\mathcal{I}_d(T_0 + d)\end{aligned}$$

Therefore, $\mathcal{I}_d(T_0 + d) \geq \gamma_1$ for $T_0 \in [0, T^*]$ and hence $\mathcal{I}_d(T_0) \geq \gamma_1$ for $T_0 \in [d, T^*]$. From the definition of function $\mathcal{F}_0$ in (4), we have $\theta(t) \geq \theta_1$ for $t \in [T, T^*]$. The contradiction is given by (20) and (21): $\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(T^*) \leq \gamma$.

Finally, for $t \geq T_0$, using (19)

$$\begin{aligned}\left\|\varepsilon(t)\right\|^2 &\leq (2\theta_1)^{2n-2}\left\|\tilde{\varepsilon}(t)\right\|^2 \\ &\leq \frac{(2\theta_1)^{2n-2}}{\alpha}\tilde{\varepsilon}'\tilde{S}\tilde{\varepsilon}(t) \\ &\leq \varepsilon^* e^{-\alpha q_m (t - T_0)} \leq \varepsilon^* e^{-\alpha q_m (t - T^*)}\end{aligned}$$

which proves the theorem.

Remark 7 *Let us observe that disturbances can be detected in a time less than d. Suppose that a perturbation occurs at time t_0. Since innovation is, in practice, computed in a sliding horizon procedure, and since Lemma 4 is valid for any $0 < \tilde{d} < d$, then innovation at times $t_0 + \tilde{d}$ contains information on that perturbation. Provided the perturbation is large enough, it is detected, and the adaptation is triggered, in a time less than d (i.e. before $t_0 + d$).*

5 Illustrative Example

In order to illustrate the performance of the *adaptive high-gain extended Kalman filter*, we introduce the system:

$$(\Sigma)\left\{\begin{array}{ccc}\begin{pmatrix}\frac{dx_1}{dt} \\ \frac{dx_2}{dt}\end{pmatrix} & = & \begin{pmatrix} x_1 - \frac{x_1^3}{3} - x_2 u \\ \epsilon\left[3sin(x_1) - x_2 - \eta\right]\end{pmatrix} \\ y & = & x_1.\end{array}\right. \qquad (22)$$

It is a modified version of the Fitzhugh-Nagumo model used for biological simulations of nerve fibers (see [6, 19] and references herein). Notice that the A matrix is input driven.
This system is trivially observable since it is already under the normal form. This is convenient since the search for normal forms is not the object of the present article. However, readers interested in this topic can refer to [3, 6, 13, 18] for instance.
The parameter η is supposed to be poorly known, or subject to sudden changes during runtime. Since it is observable, we can estimate it. We augment the state space with the variable $x_3 = \eta$, and the simple model $\dot{x_3} = 0$.

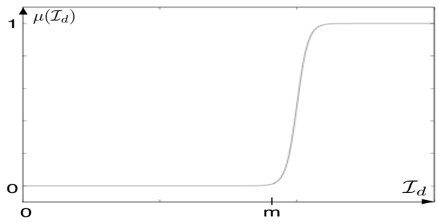

Figure 1. A sigmoid, or switch like, function.

The simulation of (Σ) and of the associated observer is performed with Matlab/Simulink. System parameters are set to $\epsilon = 0.8$, $\eta = 5$, $x_1(0) = 1.06$, and $x_2(0) = 2.69$. The input variable is a sine wave of amplitude 1, angular frequency 1 rad/sec and no offset.
A non-measured perturbation is introduced as a step change of η from 5 to -6.
Finally, the output is corrupted by an additive Orstein-Ulhenbeck process. A detailed explanation of the implementation issues can be found in [5, 8].

In a high-gain EKF, the role of the parameters Q, R, and θ_1 is both to ensure global convergence and to limit the influence of noise. They are acting one against the other, which makes them difficult to tune. In our adaptive version, those three parameters have clearly defined roles since the adaptive strategy decouples them. However, the adaptive strategy requires the use of some extra parameters. We propose the following methodology for the tuning procedure:

(1) set the performance parameters, Q, R and θ_1;
(2) set the parameters d, λ, define a function $\mu(\mathcal{I}_d)$, and choose m (See Figure 1).

The choice of those multiple parameters is based on simulation campaigns, or on the use of real data, if accessible. The tuning methodology is justified by theorem 4 as we know that configuration exists rendering the observer efficient.

Step 1
Consider a non adaptive version of the observer: $\mathcal{F} = 0$. The matrices Q and R are chosen according to the representation (1). Noise reduction is the objective, see [15] for example.
Choose $2\theta_1$ to achieve efficient converging performance and limit overshoots (see [6, 11]).
Step 2

d**:** When d (C.f. Lemma 4) is too small, innovation isn't sufficiently large to distinguish an increase in the estimation error from the influence of noise. On the other hand, a value that is too high increases the computation time as the prediction is done on a larger time interval. The choice of d has to be made from the knowledge of the time constant of the system. A fraction (e.g. $\frac{1}{3}$ to $\frac{1}{5}$) of time constant appears to be a reasonable choice.

λ: With a high value for λ, θ decreases quickly. In practice, θ often bounces up and down after large disturbances. This is avoided by setting λ to a value that is not so high in order to give some resilience to θ. For example, take $\lambda = 1$.

μ: We arbitrarily chose a sigmoid for the function μ (see Figure 1). It is a Lipschitz switch-like function that can be easily shaped.

m : It is the most important parameter. Its role appears clearly in Figure 1: adaptation is triggered only when $\mathcal{I}_d(t) > m$. One can choose the other parameters more or less arbitrarily, without any critical effect on the adaptation performance. It is not the case with m. Indeed, if we suppose that the observer estimates perfectly the state of the system, then the output trajectory predicted during the computation of innovation is equal to the output signal without noise. As far as the output signal is corrupted by noise $v(t)$, $y_{measured}(t) = y(t-d, x(t-d), \tau) + v(\tau)$ and

$$\begin{aligned}\mathcal{I}_d(t) &= \int_{t-d}^{t} \|y(t-d, x(t-d), \tau) + v(\tau) \\ &\qquad - y(t-d, z(t-d), \tau)\|^2 d\tau \\ &= \int_{t-d}^{t} \|v(\tau)\|^2 d\tau \neq 0\end{aligned}$$

Denoting σ the standard deviation of $v(t)$, a one-sigma (empirical) rule gives $\mathcal{I}_d(t) \leq \sigma^2 d$. Therefore, $m \approx \sigma^2 d$ is an appropriate choice. Notice that although m is an important parameter, it is not difficult to tune it correctly.

We display the simulation results of two scenarios. The initial state of the observer is wrong in both cases. In the first one (Figure 2), the variable η jumps only once. In the second scenario (Figures 3 and 4), η jumps repeatedly. The real state is always plotted in black, and the estimated state in red. The top graph of Figure 3 gives an account of the noise level.
On Figure 3, we can see:

- at time 60, a medium scale perturbation illustrating how innovation catches disturbances, allowing high-gain parameter to increase and ensuring convergence,
- at time 120, a perturbation too small to lead to a value of innovation large enough to trigger θ,

On Figure 4, we see at times 30, 90, 150 and 180, that innovation bounces up and down. This is the kind of situation where having set λ to a small value is useful.

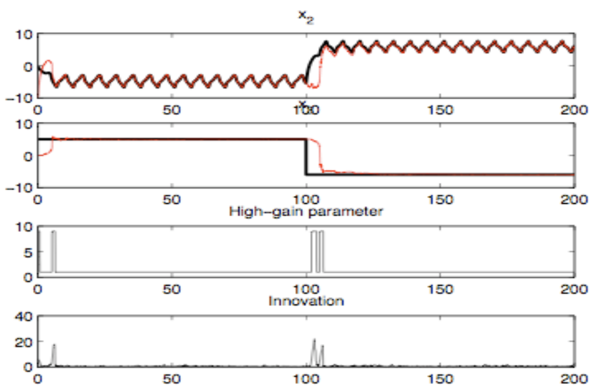

Figure 2. Simulation result of the initial scenario

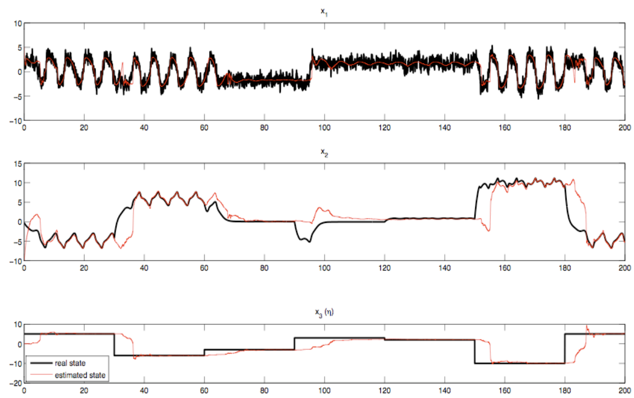

Figure 3. Second scenario: state variables estimation

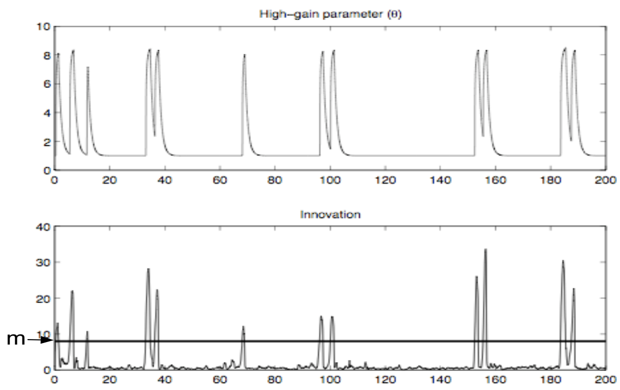

Figure 4. Second scenario: innovation and high-gain parameter

The overall behavior of the observer is the one we were searching for: noise smoothing when estimation error is small, and high-gain dynamics when large estimation error is detected. Keep in mind that an EKF, with the Q and R matrices used here, converges in 700 units of time after a jump like the one of Figure 2.

6 Conclusion

In this article we proposed an extended Kalman filter having an adaptive high-gain parameter that increases or decreases in terms of the variation of the innovation. The effect of this adaptation is that the observer mainly commutes between two modes:

- Kalman filtering mode when the innovation is small,
- High-gain mode when the innovation is large.

We proved the global exponential convergence to zero of the estimation error of the observer. We proposed guidelines for the tuning of the parameters of the observer and we performed certain simulations on a Fitzhugh-Nagumo-like model with noise and large perturbations.

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