ON CERTAIN HYPERELLIPTIC SIGNALS THAT ARE NATURAL CONTROLS FOR NONHOLONOMIC MOTION PLANNING

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Dedicated to Siddhārtha Gautama

Abstract. In this paper we address the general problem of approximating, in a certain optimal way, non admissible motions of a kinematic system with nonholonomic constraints. Since this kind of problems falls into the general subriemannian geometric setting, it is natural to consider optimality in the sense of approximating by means of subriemannian geodesics. We consider systems modeled by a subriemannian Goursat structure, a particular case being the well known system of a car with trailers, along with the associated parallel parking problem. Several authors approximate the successive Lie brackets by using trigonometric functions. By contrast, we show that the more natural optimal motions are related with closed hyperelliptic plane curves with a certain number of loops.

1. Introduction

The nonholonomic motion planning problem has attracted the attention of researchers for more than thirty years. It is out of the scope of this paper to provide a complete bibliography on the subject. We limit ourselves to mention a short bibliography which is related to our approach. Roughly speaking the literature can be separated in two different but complementary approaches: the one that proposes methods for motion planning based upon *typical* input signals such as constant controls, polynomial controls, trigonometric controls, etc., for instance H. Sussmann et al. [1, 2], and R. Murray et al. [3, 4]; and the one that pursues the formalization of the motion planning problem through the concepts of complexity, entropy and nilpotent approximations, for instance the pioneering work of F. Jean [5, 6, 7] and the series of papers of one of the authors [8, 9, 10, 11, 12, 13, 14].

The main idea we want to get across in this paper is that there is a *natural* class of input signals and admissible trajectories, that solves the motion planning problem in an optimal sense that shall be explained in the paper. In contrast with the extrinsic signals considered in the literature, the signals we propose are given by certain periodic hyperelliptic functions and are obtained intrinsically out of some invariants of the nilpotent approximation of the system. In the papers [8] and [9] some preliminary results were given in this direction, for systems involving Lie brackets up to order 3. Furthermore, the results in these papers strongly suggest a regular pattern for the signals, namely, the length of the Lie brackets of highest

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order, determines the number of loops of the hyperelliptic curve, concurrently it is natural to conjecture about the shape of the optimal signals in the general case.

In this paper we prove the conjecture on the shape of the optimal signals in the case of systems with two controls, that correspond to the so-called *Goursat structures*, that is, systems for which the higher order of the Lie brackets involved is exactly equal to the dimension of the state manifold plus two.

This class of systems includes the well known models for a car with trailers along with the so-called parallel parking problem. At the level of this example, and in accordance with the aforementioned literature, two points of view for the motion planning problem become apparent:

(1) The one that starts by observing that the system is exactly feedback equivalent to the Goursat system in chained form, and then by driving the system by means of extrinsic inputs, such as sinusoids.
(2) The one that reduces the problem to the successive *nilpotent approximations* of the system along the curve to approximate.

It is worth mentioning that in this case, the nilpotent approximation coincides with the Goursat system and the system is, in the subriemannian context, feedback equivalent to its nilpotent approximation, situation that does not occur in the general cases treated in our previous papers.

At this point it is clear that there is a polemic ingredient in the picture, namely, which are the best possible signals for driving these class of systems? Whereas most of the authors approximate brackets by sequences of trigonometric signals, or piecewise constant signals, our approach leads to an intrinsic natural class of control signals and state trajectories that are optimal in the sense of nonholonomic interpolation. Therefore we claim that ours are *the best signals to approximate the brackets at any order.*

The papers [8, 9] and [10] tackle the *generic* motion planning problem that involves brackets of order less or equal to three. By contrast, in the present paper we treat the highly *nongeneric* case of the Goursat flag, but for brackets of any order.

Our conclusions for brackets of order $1, 2$ and 3 are roughly summarized by the curves in figure 1, which shows, for a generic subriemannian metric, (with flag $(2, 3), (2, 3, 4)$, and $(2, 3, 5, 6)$ respectively), the projection of the motion along the non-admissible curve to be approximated, on the plane of the distribution (the plane of admissible motions) in certain coordinates called *normal coordinates*. In these normal coordinates, the motion for one bracket is a circle, the curve for two brackets is the well-known periodic elastica, see for instance [15], and the one for three brackets is a closed three loops universal hyperelliptic curve. A kinematic example associated to this last hyperelliptic curve corresponds to the motion planning of *the ball with a trailer* discussed in [9]. The interested reader is invited to see the animation showing this motion in `http://www.lsis.org/boizotn/KinematicVids/`.

Another interesting picture showed in figures 2 and 4 is a peanut shape curve that we shall call the *cacahuète*, the vertical coordinate is the first control and the horizontal coordinate is one of the normal coordinates in the distribution.

It is completely natural to conjecture that the series depicted by the cacahuètes and the corresponding closed multi loops curves persists. Although our intuition points out in that direction, up to now it is not clear if such persistence remains in general. The aim of this paper is to show in a constructive way, that for Goursat

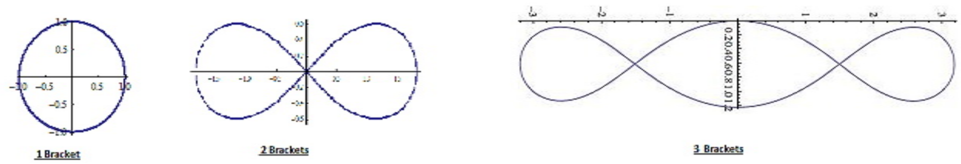

FIGURE 1. Generic optimal interpolating motion for 1 to 3 brackets.

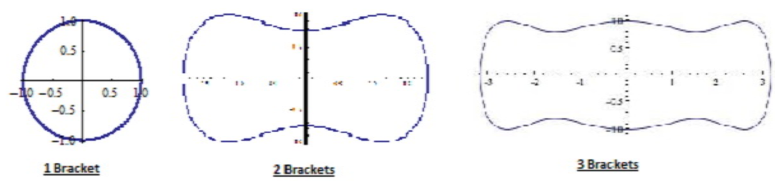

FIGURE 2. Cacahuètes for 1 to 3 brackets

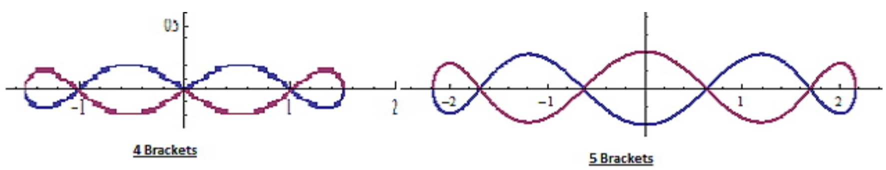

FIGURE 3. Optimal interpolating motion for Goursat structure with 4 and 5 brackets

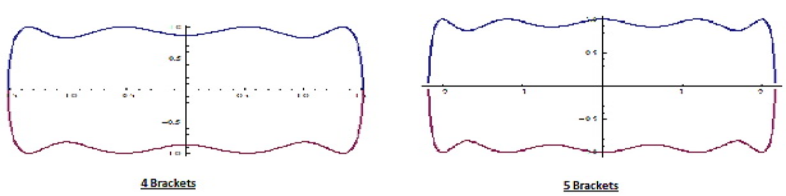

FIGURE 4. Cacahuètes for 4 and 5 brackets

structures this series continues whatever the number of brackets. The optimal motion is shown, in certain coordinates adapted to the Goursat structure, in figure 3 for Lie brackets of order 4 and 5.

The organization of the paper is as follows: in section 2 we set our notations, give the precise statement of the problem and summarize some preliminary results from the papers [8, 9, 10] and [11] in connexion with generic cases of small coranks. We also introduce special coordinates adapted to the problem and reformulate, for the Goursat case, a result that allows to reduce the interpolation entropy to the nilpotent approximation. At the end of the section, we present our main result along with some remarks on the interpolation entropy methodology that are important in applications. In section 3 we show that there is essentially only one left invariant

Goursat subriemannian structure, and complete the proof of the main result by dealing with the integrability of Goursat structures, following the integration process developed in [16]. In section 4 some explicit computations in coordinates for optimal curves are carried out. In section 5, using the integrability results and the explicit formulæ for extremals, we show that the conjecture about the persistence of the geometric pattern of optimal trajectories certainly holds in the Goursat case. In Section 6, we derive some conclusions and pose two challenging questions that still remain open.

2. The Goursat motion planning problem

In this section we present the notations used in the paper, the summary of the known results for the generic case, the description of the Goursat motion planning, the statement of the problem and our results.

2.1. **Notations and preliminary results.**

We present in this paragraph the general lines of the subriemannian entropy interpolation theory, for details we refer the reader to the series of papers [8, 9, 10, 11, 12]. As customary, we work in the smooth category and the genericity we refer to is with respect to the Whitney topology.

A rank-2 subriemannian metric over a n-dimensional manifold M is a pair (Δ, g) where Δ is a 2-dimensional vector-distribution on M, and g is a Riemannian metric over Δ. Equivalently, the metric is specified by the following control system over M

$$(\Sigma) \quad \dot{x} = F_1(x)u_1 + F_2(x)u_2, \tag{2.1}$$

in such a way that the vector fields F_1 and F_2 form an orthonormal frame for g. Geodesic curves (length minimizing curves) are those which minimize the functional

$$C_1(u) = \int_0^T \sqrt{(u_1(t))^2 + (u_2(t))^2}\ dt, \tag{2.2}$$

in free time, or equivalently the functional

$$C_2(u) = \int_0^T ((u_1(t))^2 + (u_2(t))^2)\ dt,$$

in fixed time T.

The *interpolation entropy* of a path $\Gamma \colon [0, T_\Gamma] \to M$ transversal to the distribution Δ was introduced by J.P. Gauthier and V. Zakalyukin in [10], following the pioneering work of F. Jean in [6]; this concept is related to the Kolmogorov entropy of a path, considering it as a metric space with respect to the subriemannian metric.

> For any $\varepsilon > 0$ consider $\ell(\varepsilon)$ the minimum subriemannian length of a $\Sigma-$admissible curve that interpolates Γ by means of pieces of sub-riemannian length $\leq \varepsilon$, the function $\ell(\varepsilon)$ tends to infinity when ε tends to zero. The $\varepsilon-$*interpolation entropy* of Γ, denoted as $E_\Gamma^\Sigma(\varepsilon)$, is the leading term of $\varepsilon^{-1}\ell(\varepsilon)$, (modulo the equivalence relation $\ell_1(\varepsilon) \approx \ell_2(\varepsilon)$ if and only if $\lim_{\varepsilon\to 0} \frac{\ell_1(\varepsilon)}{\ell_2(\varepsilon)} = 1$).

For a generic pair (Σ, Γ) with Δ a p-step bracket generating distribution and Γ a transversal path to Δ, the entropy has an expression of the form $E_\Gamma^\Sigma(\varepsilon) = \frac{a}{\varepsilon^p}$. However this is true only in the absence of codimension 1 generic singularities, in which case one has $E_\Gamma^\Sigma(\varepsilon) = -\ln(\varepsilon)\frac{a}{\varepsilon^p}$. The generic expression of the constants a has been exhausted for small values of the ranks and co-ranks. For details we refer the reader to the aforementioned papers where proofs are always constructive.

Approaching the motion planning problem (Σ, Γ) by means of the subriemannian formalism and the interpolation entropy of Γ provides a natural optimal way to approximate Lie brackets, issue that is fundamental in control theory, and that has been addressed by a number of people using trigonometric control signals u_1 and u_2 for the approximation. We claim that sinusoids are pertinent for first order Lie brackets only, but far from being optimal for higher order Lie brackets. In the recent work [17], which has strong connections with the present paper, we argue about the use of Lissajous like controls for approximating high order Lie brackets in the free nilpotent Lie algebra associated with F_1 and F_2.

Given a generic pair (Σ, Γ), the behavior of the system along Γ is dominated by what we call *nilpotent approximation of the system along* Γ, roughly speaking it can be viewed as a one parameter family of nilpotent approximations of the system at the points of Γ. For the concept of nilpotent approximation, we refer the reader to A. Bellaïche [18].

The standard concept of nilpotent approximation at a point relies on the so-called *adapted coordinates*, that can be obtained by polynomial transformations from any coordinate system, while our concept of nilpotent approximation along a curve relies on *normal coordinates* which are the subriemannian analog of the normal coordinates in Riemannian geometry. In normal coordinates, the curve Γ is rectified to become a *vertical* line given by the last coordinate, whereas the distribution along Γ is realized by the *horizontal* plane given by the first two coordinates. The results in the rank 2 case are summarized in the following:

Theorem 1. *(Rank 2, generic couple (Σ, Γ)). The curves that minimize the entropy (maximize the displacement in the direction of Γ) are (small modifications of) those that minimize the entropy of the nilpotent approximation along Γ. The horizontal projections of these entropy-minimizing curves are the following closed plane curves (shown in figure 1):*

***case 1**: corank 1 ($n = 3$, Dubins car) circle,*
***case 2**: corank 2 ($n = 4$, car with a trailer) periodic elastica,*
corank 3 ($n = 5$, ball rolling on a plane) periodic elastica,
***case 3**: corank 4 ($n = 6$, ball with a trailer) 3-loops hyperelliptic curve.*

Remark 1. *In all cases, the curves are universal once normalized to have length one, although they should be of length ε. This asymptotic result is obtained for $\varepsilon \to 0$, and is dominated by the nilpotent approximation which is defined out of a gradation of the involved variables and their corresponding derivatives, and consequently has a certain quasi-homogeneity property (with respect to the gradation). This property allows to obtain the ε-optimal curves by means of quasi-homogeneous dilations from the curves described in the theorem.*

Remark 2. *In case 3 we were not able to show the result in the general corank 4 case. We proved it only in the case of (the nilpotent approximation of) the ball with* *a trailer. Surprisingly in that case the system that yields the geodesics is Liouville-integrable, while in the general corank 4 case it is not.*

2.2. **The Goursat case.**

2.2.1. *Preliminaries.* This section presents the main result of the paper on the non-holonomic interpolation problem on Goursat structures.

Associated with a 2-dimensional (in fact, to any) distribution Δ on a n dimensional manifold M, the following two flags of distributions

$$\begin{aligned} \Delta_i : &\quad \Delta_0 = \Delta,\ \Delta_{i+1} = [\Delta, \Delta_i], \\ D_i : &\quad D_0 = \Delta,\ D_{i+1} = [D_i, D_i] \end{aligned}$$

are well defined.

We assume that pair (Δ, Γ) with Δ a bracket generating distribution and Γ a transversal curve to Δ is given once for all. Such a pair shall be called *Goursat pair* if all the distributions D_i, Δ_i have rank strictly less than n, and are minimal-equiregular in the following sense: all the distributions D_i, Δ_i have constant rank r_i along Γ with $r_0 = 2, r_1 = 3, r_{i+1} = r_i + 1$ as long as $r_i \leq n$. A distribution Δ is usually called Goursat distribution if satisfies the minimal-equiregularity condition in all M.

Goursat distributions have been in the literature for long, apparently they were introduced by E. von Weber [19] at the beginning of the last century within the general framework of differential equations. This class of distributions has been incorporated into the geometric control theory literature partly because it provides good models for the study of certain kinematic systems, such as the one of the car with trailers.

Proposition 1. [19] *The dimension n being fixed, there is a single Goursat distribution, up to a (local) diffeomeorphism, generated by the two following (local) vector fields:*

$$\begin{cases} X_1 = \dfrac{\partial}{\partial x_1}, \\ X_2 = \displaystyle\sum_{i=2}^{n} \frac{(x_1)^{i-2}}{(i-2)!} \frac{\partial}{\partial x_i}. \end{cases} \tag{2.3}$$

This result endows $M = \mathbb{R}^n$ with the structure of $(n-1)-$step nilpotent Lie algebra, by considering the generating relations:

$$X_i = [X_1, X_{i-1}],\ i = 3, \dots, n \tag{2.4}$$

as the only non-vanishing Lie brackets. The exponential mapping establishes a local diffeomorphism onto a $(n-1)-$step nilpotent simply connected Lie group, that we shall call the *Goursat group.*

Remark 3. *The normal form (2.3) is the not the usual one that commonly appears in the literature, but is shown in* [16] *that they are diffeomorphic.*

Remark 4. *It is clear that the problem of parallel parking of a car with $n-3$ trailers reduces to the problem of ε-approximating (or ε-interpolating) the trajectory* *of the vector field $X_n = [X_1, X_{n-1}]$, that is, the x_n-axis, by means of an admissible trajectory.*

A Goursat motion planning problem on a n dimensional manifold M is a triple $\mathcal{G} = (\Delta, g, \Gamma)$ where (Δ, g) is a subriemannian structure on M and (Δ, Γ) is a Goursat pair. A one parameter family of admissible curves γ_ε realizing the interpolation entropy of the curve Γ is in a sense the optimal way to approximate Γ by ε-close admissible curves.

In this paper we give a general solution to the Goursat motion planning, and show that the solution generalizes the generic cases described in Theorem 1. Since everything is local and because of Proposition 1, there is no loss of generality in assuming that $M = \mathbb{R}^n$ with coordinates $x = (x_1, \ldots, x_n)$. We shall proceed in three steps:

(1) We define and compute the nilpotent approximation of $\mathcal{G}$ along Γ.
(2) We show that computing the entropy reduces to computing that of the nilpotent approximation.
(3) We solve the corresponding optimal control problem, which is possible due to Liouville integrability of the corresponding Hamiltonian system.

The two first steps are carried out in the two next paragraphs.

2.2.2. *Nilpotent approximation of $\mathcal{G}$ along Γ.* The data $\mathcal{G} = (\Delta, g, \Gamma)$ is given, with $\Delta = \operatorname{span}\{F_1, F_2\}$ a Goursat distribution and Γ a smooth curve tranversal to Δ. The vector fields F_1, F_2 define the control system (2.1), that we shall write simply as $\Sigma = (F1, F2)$.

Σ is feedback equivalent to the normal form (2.3) therefore there exist functions $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma$ nowhere vanishing, and local coordinates $x = (x_1, \ldots, x_n)$, that we shall call *Goursat coordinates*, such that

$$F_1(x) = \alpha(x)X_1(x) + \beta(x)X_2(x),$$
$$F_2(x) = \gamma(x)X_1(x) + \delta(x)X_2(x).$$

Generically the curve $\Gamma : [0, S] \to \mathbb{R}^n,\ s \mapsto \Gamma(s) = (\gamma_1(s), \ldots, \gamma_n(s))$ is transversal to $\Delta_{n-3} = D_{n-3}$, therefore $\gamma_n'(s) \neq 0$ and we can make the following change of coordinates:

$$\text{(2.5)} \qquad \begin{cases} \tilde{x}_i = x_i - \gamma_i \circ \gamma_n^{-1}(x_n), \quad \text{for} \quad i = 1, \ldots, n-1, \quad \text{and} \\ \tilde{x}_n = x_n, \end{cases}$$

in these new coordinates the curve Γ is *rectified* to become the *vertical* line $\widetilde{\Gamma}(s) = (0, \ldots, 0, \gamma_n(s))$.

In what follows, we consider that the change of coordinates has been carried out, and we shall omit the tilde symbol in both the coordinates and the curve.

For an arbitrary but fixed point $\Gamma(s)$ on the curve Γ, the coordinates $x_1, \ldots, x_{n-1}$ are centered at zero but x_n is not. Assuming that the last coordinate is small, we consider the gradation in the formal power series in the variables $x_1, \ldots, x_{n-1}, x_n - \gamma_n(s)$ obtained by assigning weight 1 to both x_1 and x_2, weight 2 to x_3, weight 3 to $x_4, \ldots$, weight $(n-2)$ to x_{n-1} and weight $(n-1)$ to $x_n - \gamma_n(s)$. This gradation induces a gradation in the formal vector fields in such way that both $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ have weight -1, $\frac{\partial}{\partial x_3}$ has weight $-2, \ldots, \frac{\partial}{\partial x_{n-1}}$ has weight $-(n-2)$ and $\frac{\partial}{\partial x_n}$ has weight $-(n-1)$. This gradation is homogeneous with respect to the Lie bracket operation along the curve Γ.

We define the *Goursat nilpotent approximation* $\widehat{\Sigma} = (\widehat{F_1}, \widehat{F_2})$ of Σ along Γ to be the homogeneous term of order -1 of the gradation, that is,

$$\begin{cases} F_1 = \alpha(\Gamma(s))X_1(x_1) + \beta(\Gamma(s))X_2(x_1) + O^0 = \widehat{F}_1\Big|_{x=\Gamma(s)} + O^0, \\ F_2 = \gamma(\Gamma(s))X_1(x_1) + \delta(\Gamma(s))X_2(x_1) + O^0 = \widehat{F}_2\Big|_{x=\Gamma(s)} + O^0. \end{cases} \tag{2.6}$$

As customary, O^n denotes a vector field of order n. A function of order n is a function contained in the n^{th} power of the ideal generated by $x_1, \ldots, x_{n-1}, x_n - \gamma_n(s)$.

For both, the system Σ and its nilpotent approximation $\widehat{\Sigma}$, we have the following rough estimates, for admissible curves $x : [0,T] \to \mathbb{R}^n,\ t \mapsto (x_1(t)x_2(t), \ldots, x_n(t))$ with initial point on Γ $(x(0) \in \Gamma)$, and (subriemannian) length
$l(\varepsilon) = \int_0^T \sqrt{(u_1(t))^2 + (u_2(t))^2} \leq \varepsilon$:

$$|x_i(t)| \leq k\varepsilon, \quad i = 1, 2; \tag{2.7}$$

$$|x_i(t)| \leq k\varepsilon^{i-1}, \quad i = 3, n-1;$$

$$|x_n(t) - x_n(0)| \leq k\varepsilon^{n-1}, \tag{2.8}$$

with an appropriate constant k.

2.2.3. *Equivalence between entropy of the system and that of its nilpotent approximation.* Since the estimates in this paragraph are similar to those of the generic cases of small coranks, we shall give here only the general lines of the proof, the interested reader can see more details in [10].

For the same control functions $u_1(t), u_2(t)$ we consider two admissible curves $x(t)$ and $\widehat{x}(t),\ t \in [0,T]$ of the systems Σ and and $\widehat{\Sigma}$ respectively. Further we assume that both curves have length $\leq \varepsilon$ and the same initial point $x(0) = \widehat{x}(0) \in \Gamma$. Set $e(t) = x(t) - \widehat{x}(t)$

First for $i = 1, 2$ we have

$$\dot{e}_i(t) = O^1,$$

therefore

$$|e_i(t)| \leq k\varepsilon^2,\ i = 1, 2;$$

now for $i > 2$ we have

$$|\dot{e}_i(t)| \leq \frac{|(x_1)^{i-2} - (\widehat{x}_1)^{i-2}|}{(i-2)!} + O^{i-1},$$

as a consequence, taking into account (2.7), we have that

$$|e_i(t)| \leq k\varepsilon^i,\ i = 3, \ldots, n,$$

of course there is no loss of generality assuming that the constant k is the same as in the rough estimates (2.7).

This means that the subriemannian distance between $x(t)$ and $\widehat{x}(t)$ relative to either Σ or $\widehat{\Sigma}$, is smaller than $k\varepsilon^{(1+\frac{1}{n-1})}$, that is,

$$d(x(t), \widehat{x}(t)) \leq k\varepsilon^{(1+\frac{1}{n-1})}. \tag{2.9}$$

With these estimates in hand, we can choose now a curve $x(t) : [0, \theta] \to \mathbb{R}^n$ that $\varepsilon-$interpolates Γ by maximizing $x_n(\theta)$, and applying the previous inequality to the interpolating pieces, we get that

$$E_\Gamma^{\widehat{\Sigma}}(\varepsilon) \leq E_\Gamma^{\Sigma}(\varepsilon). \tag{2.10}$$

In this line of argumentation we can interchange the role of x and $\widehat{x}$ to get finally that:

$$E_\Gamma^{\widehat{\Sigma}}(\varepsilon) \approx E_\Gamma^{\Sigma}(\varepsilon) \tag{2.11}$$

2.3. **Statement of the main result.** Before stating the theorem, it is worth pointing out that our result can be understood and utilized in practical applications of motion planning, under two completely different viewpoints:

- One may think that the Goursat system is given, the car with trailers for instance, and that it is put via feedback and change of coordinates under the canonical form (2.3). After that, one chooses to apply the interpolation entropy strategy. This procedure shall provide an exact ε-interpolation control strategy, but with a non-natural cost (due to the preliminary feedback). It is equivalent, after feedback, to solve the problem of finding admissible ε-interpolating curves that have an arbitrary but fixed subriemannian length given by (2.2), and that maximize the distance on Γ between two successive interpolated points. Under this strategy the actual size of ε is irrelevant.
- The other viewpoint is to consider *any* subriemannian metric whatsoever over the Goursat distribution, and to apply without preliminary feedback, the interpolation entropy strategy for small ε. In such a case, the metric is free, but the result is asymptotic only (when $\varepsilon \to 0$). It is noticeable that the (asymptotic) result is in fact independent of the chosen metric.

Our result is the following:

Theorem 2. *(In Goursat coordinates, for model 2.6 or for its nilpotent approximation $\widehat{\Sigma}$.) A length one extremal of the interpolation problem between the origin and a point of the x_n axis, maximizing the endpoint coordinate x_n can be explicitly calculated. Its projection on the plane (x_1, x_2) is a closed hyperelliptic curve, smooth-periodic, with $n-2$ loops, shown on figure 3 for $n = 6, 7$. The curve that interpolates with length ε is obtained from this one by homogeneity.*

According to the estimates in paragraphs 2.2.2 and 2.2.3, what remains to be done is to find the explicit equations for the extremals along with the corresponding projections. The key point for proving this result is the fact that the Hamiltonian system of geodesic equations of the Goursat model is Liouville integrable whatever the dimension n. We carry this out in the next section, following the integration process developed in [16].

3. Extremal curves for the Goursat Case

3.1. The optimal control problem. As explained in the paragraph 2.1, in general the subriemannian geodesic problem on a subriemannian manifold is tantamount to an optimal control problem with quadratic cost. For our data $\mathcal{G} = (\Delta, g, \Gamma)$ we deal with a left invariant optimal control problem on the Goursat Lie group G. This Lie group can be identified with $\mathbb{R}^n$ with the group law defined as follows, if $g_1 = (x_1, \dots, x_n)$ and $g_2 = (y_1, \dots, y_n)$ are two elements in G, then $g_1 \oplus g_2 = (z_1, \dots, z_n)$ with:

$$\begin{cases} z_1 = x_1 + y_1, \\ z_2 = x_2 + y_2, \quad \text{and} \\ z_k = x_k + y_k + \displaystyle\sum_{j=2}^{k-1} \frac{x_1^{k-j}}{(k-j)!}\, y_j, \quad \text{for} \quad k = 3, \dots, n, \end{cases} \tag{3.1}$$

and the group identity $e = (0, \dots, 0)$.
The vector fields in the normal form (2.3) are $\oplus-$left invariant, and, as mentioned before, G is a simply connected $n-$step nilpotent Lie group with Lie algebra $\mathfrak{g}$, whose basis is defined by means of the commuting relations (2.4).

At the level of the nilpotent approximation of the left invariant control problem we deal with is the following: among the admissible trajectories of the system

$$\dot{g} = u_1(\alpha X_1(g) + \beta X_2(g)) + u_2(\gamma X_1(g) + \delta X_2(g)), \quad g \in G, \tag{3.2}$$

with $\alpha\delta - \beta\gamma \neq 0$, find the one that minimizes

$$\int (u_1(t)^2 + u_2(t)^2)\; dt. \tag{3.3}$$

As it is explained in the papers ([8, 9, 10]), we have to find a minimizing geodesic of length 1 (homogeneity) connecting the origin to a point on the x_n axis.

3.2. Gauge invariance of the metric. Any arbitrary nonsingular 2×2 matrix

$$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix},$$

can be written as $A = TR_\varphi$ where $R_\varphi \in \mathrm{SO}_2$ and T is a non-singular lower triangular 2×2 matrix, say

$$T = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \quad \text{with} \quad ac \neq 0.$$

Since the rotation R_φ preserves the metric it is enough to consider the constant transformation given by T, in such a case the system (3.2) writes as follows

$$\dot{g} = (aX_1(g) + bX_2(g))\; u_1 + cX_2(g)\; u_2 \tag{3.4}$$

and in coordinates we have

$$\begin{cases} \dot{x}_1 = au_1, \\ \dot{x}_2 = bu_1 + cu_2, \quad \text{and} \\ \dot{x}_k = b\dfrac{x_1^{k-2}}{(k-2)!}u_1 + c\dfrac{x_1^{k-2}}{(k-2)!}u_2, \quad \text{for} \quad k = 3, \dots, n. \end{cases} \tag{3.5}$$

We consider the change of coordinates

$$g = (x_1, \dots, x_n) \mapsto \widehat{g} = (\widehat{x}_1, \dots, \widehat{x}_n),$$

given by

$$\begin{cases} \widehat{x}_1 = \dfrac{1}{a}x_1, \\ \widehat{x}_2 = -\dfrac{b}{ca}x_1 + \dfrac{1}{c}x_2, \quad \text{and} \\ \widehat{x}_k = -\dfrac{b}{ca^{k-1}}\dfrac{x_1^{k-1}}{(k-1)!} + \dfrac{1}{ca^{k-2}}x_k \quad \text{for} \quad k = 3, \dots, n, \end{cases} \tag{3.6}$$

a straightforward computation yields

$$\dot{\widehat{g}} = u_1 X_2(\widehat{g}) + u_2 X_2(\widehat{g}), \tag{3.7}$$

as can be easily verified.

Furthermore the left invariance of the vector fields X_1 and X_2 implies that for any $h \in G$ we have

$$\langle DL_h(\dot{\widehat{g}}), DL_h(\dot{\widehat{g}})\rangle = \langle \dot{\widehat{g}}, \dot{\widehat{g}}\rangle = u_1^2 + u_2^2, \tag{3.8}$$

where $h \mapsto L_h$ is the left translation.

We summarize the result:

Theorem 3. *Given a Goursat triple $\mathcal{G} = (\Delta, g, \Gamma)$, we can choose coordinates along Γ and orthonormal vector fields F_1, F_2 generating Δ, such that the nilpotent approximation along Γ writes*

$$\dot{x} = X_1(x)u_1 + X_2(x)u_2,$$

and $\Gamma(s) = (0, \dots, 0, \varphi(s))$ for some smooth function $\varphi(s)$.

Lemma 1. *The map $: G \to G,\ g \mapsto \widehat{g}$ is an automorphism of G.*

Proof. From the definition (3.6) it is clear that $g \mapsto \widehat{g}$ is a diffeomorphism. Let $g_1 = (x_1, \dots, x_n)$ and $g_2 = (y_1, \dots, y_n)$ be two arbitrary elements in G and let $g_1 \oplus g_2 = (z_1, \dots z_n)$. If we denote as $\widehat{g}_1 = (\widehat{x}_1, \dots, \widehat{x}_n), \widehat{g}_2 = (\widehat{y}_1, \dots, \widehat{y}_n)$ and $\widehat{g_1 \oplus g_2} = (\widehat{z}_1, \dots \widehat{z}_n)$ the corresponding images, then the first two coordinates of $\widehat{g}_1 \oplus \widehat{g}_2$ are given by

$$\begin{aligned} \widehat{x}_1 + \widehat{y}_1 &= \frac{1}{a}x_1 + \frac{1}{a}y_1 = \widehat{z}_1, \\ \widehat{x}_2 + \widehat{y}_2 &= \left(-\frac{b}{ca}x_1 + \frac{1}{c}x_2\right) + \left(-\frac{b}{ca}y_1 + \frac{1}{c}y_2\right) = \widehat{z}_2, \end{aligned}$$

whereas for its k^{th} coordinate, with $k = 3, \ldots, n$ we have

$$
\begin{aligned}
\widehat{x}_k + \widehat{y}_k + \sum_{j=2}^{k-1} \frac{\widehat{x}_1^{\,k-j}}{(k-j)!}\widehat{y}_j &= -\frac{b}{ca^{k-1}}\frac{x_1^{k-1}}{(k-1)!} + \frac{1}{ca^{k-2}}x_k \\
&\quad -\frac{b}{ca^{k-1}}\frac{y_1^{k-1}}{(k-1)!} + \frac{1}{ca^{k-2}}y_k \\
&\quad +\sum_{j=2}^{k-1}\frac{\left(\frac{x_1}{a}\right)^{k-j}}{(k-j)!}\left[-\frac{b}{ca^{j-1}}\frac{y_1^{j-1}}{(j-1)!} + \frac{1}{ca^{j-2}}y_j\right] \\
&= -\frac{b}{ca^{k-1}}\left[\frac{x_1^{k-1}}{(k-1)!} + \sum_{j=2}^{k-1}\frac{x_1^{k-j}}{(k-j)!}\frac{y_1^{j-1}}{(j-1)!} + \frac{y_1^{k-1}}{(k-1)!}\right] \\
&\quad +\frac{1}{ca^{k-2}}\left[x_k + y_k + \sum_{j=2}^{k-1}\frac{x_1^{k-j}}{(k-j)!}y_j\right] \\
&= -\frac{b}{ca^{k-1}}\frac{(x_1+y_1)^{k-1}}{(k-1)!} \\
&\quad +\frac{1}{ca^{k-2}}\left[x_k + y_k + \sum_{j=2}^{k-1}\frac{x_1^{k-j}}{(k-j)!}y_j\right] \\
&= \widehat{z}_k.
\end{aligned}
$$

It follows that $\widehat{g_1 \oplus g_2} = \widehat{g}_1 \oplus \widehat{g}_2$

□

Now, a (weak) corollary of Theorem (3) and Lemma (1) is the following

Corollary 1. *In the Goursat group, up to automorphisms, there is only one sub-riemannian left invariant metric that is defined by the normal form (2.3) with X_1 and X_2 orthonormal.*

3.3. **The entropy.** Going back to the motion planning problem, we observe first that the automorphism (3.6) essentially does not change the reference path since now we have

$$s \mapsto \Gamma(s) = (0, \ldots, 0, \frac{1}{ca^{n-2}}\gamma_n(s)),$$

and a formula for the entropy can be explicitly written.

In fact, if ω is the 1−form that vanishes on the distribution $\Delta^{(n-2)}$ and which satisfies $\omega(X_{n-1}) = 1$ where $X_{n-1} = [X_1, X_{n-2}]$, then by denoting $f(s) = \omega(\dot{\Gamma}(s))$ we have

$$\widehat{E}(\varepsilon) \approx E(\varepsilon) \approx \int_\Gamma f(s)\ ds \cdot \frac{A_n}{\varepsilon^n} \tag{3.9}$$

where A_n is a universal constant that depends on the dimension n only.

3.4. **Application of the Pontryagin Maximum Principle.** We now tackle the left invariant optimal control problem on the Goursat group G consisting of the minimization of the functional

$$\int u_1^2 + u_2^2, \tag{3.10}$$

among the admissible trajectories of the left invariant control system

$$\dot{g} = u_1 X_1(g) + u_2 X_2(g), \tag{3.11}$$

with X_1 and X_2 given by the normal form (2.3) and satisfying the commuting relations (2.4), the Lie algebra of G is denoted as $\mathfrak{g}$ and can be identified with T_eG.

As it is well known, Pontryagin maximum principle provides a standard geometric tool for the description of extremals by establishing necessary conditions for optimality, for details we refer the reader to ([20]).

If p denotes the dual variable in $\mathfrak{g}^*$, then for each vector field X_i we have the corresponding Hamiltonian $H_i = \langle p, X_i \rangle, i = 1, \ldots n$ with Poisson brackets satisfying commuting relations dual to those of (2.4), that is,

$$H_i = \{H_1, H_{i-1}\}, \quad i = 3, \ldots n. \tag{3.12}$$

Maximality condition of the Pontryagin Maximum Principle yields

$$u_1 = H_1, \quad \text{and} \quad u_2 = H_2,$$

and the system Hamiltonian becomes quadratic

$$\mathcal{H} = \frac{1}{2}(H_1^2 + H_2^2),$$

the associated adjoint equations are obtained by differentiating along the extremal as customary: $\dot{H}_i = \{H_i, \mathcal{H}\}$. In consequence, the commuting relations (3.12) clearly yield

$$\dot{H}_1 = H_2 H_3 \tag{3.13}$$

$$\dot{H}_2 = -H_1 H_3 \tag{3.14}$$

$$\dot{H}_3 = -H_1 H_4 \tag{3.15}$$

$$\vdots$$

$$\dot{H}_i = -H_1 H_{i+1} \tag{3.16}$$

$$\vdots$$

$$\dot{H}_{n-3} = -H_1 H_{n-2} \tag{3.17}$$

$$\dot{H}_{n-2} = -H_1 H_{n-1} \tag{3.18}$$

$$\dot{H}_{n-1} = -H_1 H_n \tag{3.19}$$

$$\dot{H}_n = 0 \tag{3.20}$$

Therefore H_n is constant along extremals and shall be denoted

$$c_1 := H_n = H_n(0).$$

3.5. **Geometry of extremals.** From equations (3.19) and (3.18) one gets

$$\frac{d}{dt}\left(\frac{1}{2!}H_{n-1}^2 - c_1 H_{n-2}\right) = 0$$

Another conservation law is then obtained as

$$c_2 := \frac{1}{2!}H_{n-1}^2(0) - c_1 H_{n-2}(0)$$

Now starting from the polynomial equation in the variables (H_{n-1}, H_{n-2})

$$\frac{1}{2!}H_{n-1}^2 - c_1 H_{n-2} - c_2 = 0, \tag{3.21}$$

and using the adjoint system, a further conservation law can be generated by multiplying for $\dot{H}_{n-1}$:

$$\frac{1}{2!}H_{n-1}^2 \dot{H}_{n-1} - c_1 H_{n-2}\dot{H}_{n-1} - c_2 \dot{H}_{n-1} = 0,$$

for then equations (3.18) and (3.17) imply

$$\frac{d}{dt}\left(\frac{1}{3!}H_{n-1}^3 - c_1^2 H_{n-3} - c_2 H_{n-1}\right) = 0.$$

In consequence, another conservation law c_3 is obtained and the procedure can continue with a polynomial equation in the variables (H_{n-1}, H_{n-3})

$$\frac{1}{3!}H_{n-1}^3 - c_1^2 H_{n-3} - c_2 H_{n-1} - c_3 = 0.$$

Following this procedure of multiplying by $\dot{H}_{n-1}$ and integrating, an extra conservation law is obtained at each step j, together with a polynomial equation in the variables (H_{n-1}, H_{n-j}) with $j = 2, \ldots, k$. For instance, next step yields c_4 together with the polynomial equation in the variables (H_{n-1}, H_{n-4})

$$\frac{1}{4!}H_{n-1}^4 - c_1^3 H_{n-4} - \frac{c_2}{2!}H_{n-1}^2 - c_3 H_{n-1} - c_4 = 0.$$

The last two steps of this process make use of equations (3.15) and (3.14) and (3.14) and (3.13) respectively. By performing the corresponding integration the conservation laws c_{k-1} and c_k are obtained, together with the polynomial equations in the variables (H_{n-1}, H_3) and (H_{n-1}, H_2)

$$\frac{1}{(k-1)!}H_{n-1}^{k-1} - c_1^{k-2} H_3 - \sum_{j=2}^{k-1} \frac{c_j}{(k-1-j)!} H_{n-1}^{k-1-j} = 0 \text{ and} \tag{3.22}$$

$$\frac{1}{k!}H_{n-1}^{k} - c_1^{k-1} H_2 - \sum_{j=2}^{k} \frac{c_j}{(k-j)!} H_{n-1}^{k-j} = 0, \tag{3.23}$$

respectively. In conclusion we have the following

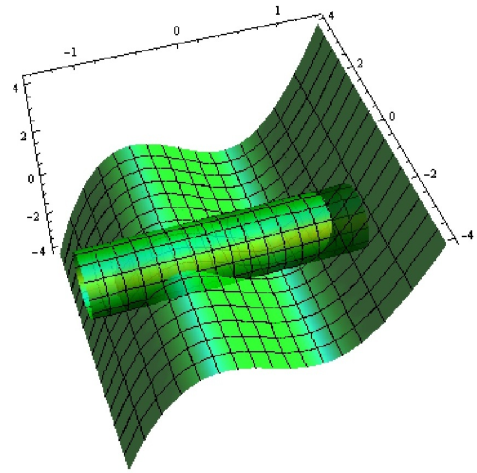

FIGURE 5. Intersection of the energy cylinder $\mathcal{H} = \frac{1}{2}$ and the cylinder with generatrix (3.23).

Proposition 2. *The adjoint system given by equations (3.13) to (3.20) is Liouville integrable and the extremal curves lie in the intersection of the energy cylinder $\mathcal{H} = \frac{1}{2}$ and the cylinder with generatrix (3.23).*

This fact is very useful and it is illustrated in the figure (5), where the intersection of these two cylinders is shown.

4. Computations in coordinates

In this section we change the notation having in mind some applications, we write the dimension as $n = 2 + k$, and the coordinates as $(x, y, \theta_1, \dots, \theta_k)$. The corank k distribution is given by the normal form (2.3), that in these coordinates writes as follows:

$$X_1 = \frac{\partial}{\partial x},$$
$$X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial \theta_1} + \frac{x^2}{2!}\frac{\partial}{\partial \theta_2} + \frac{x^3}{3!}\frac{\partial}{\partial \theta_3} + \cdots + \frac{x^k}{k!}\frac{\partial}{\partial \theta_k}.$$

The Lie algebra generated by $\{X_1, X_2\}$ satisfies the commuting relations (2.4), and is isomorphic to the $(k+1)-$step nilpotent Goursat Lie algebra $\mathfrak{g}$. The control system (Σ) in these coordinates is written as follows:

$$\begin{aligned}
\dot{x} &= u_1 \\
\dot{y} &= u_2 \\
\dot{\theta}_1 &= x u_2 \\
\dot{\theta}_2 &= \frac{x^2}{2!} u_2 \\
\dot{\theta}_3 &= \frac{x^3}{3!} u_2 \\
&\vdots \\
\dot{\theta}_k &= \frac{x^k}{k!} u_2
\end{aligned}$$

The Pontryagin maximum principle applies in the same lines as before, to obtain that along extremals one has

$$\dot{x} = H_1, \quad \dot{y} = H_2 \quad \text{and} \quad \dot{\theta}_j = H_2 \frac{x^j}{j!}$$

Lemma 2. *H_{n-1} is a linear function of x. Moreover x and $\dot{y}$ are periodic functions of time.*

Proof. In fact equation (3.19) writes

$$\dot{H}_{n-1} = -c_1 H_1 = -c_1 \dot{x}.$$

In consequence:

$$H_{n-1} = H_{n-1}(0) - c_1(x - x(0)).$$

□

Without loss of generality we can assume that $c_1 = H_n = H_n(0) = -1$ and $H_{n-1}(0) = x(0) = 0$ in such a way that H_{n-1} may be identified with x.

4.1. **The hyperelliptic curve.** Following the inductive procedure explained in paragraph (3.5) and assuming that $c_1 = -1$ we can write equations (3.23) and (3.22) as follows:

$$H_3 = -\frac{1}{(k-1)!} x^{k-1} + \sum_{j=2}^{k-1} \frac{c_j}{(k-1-j)!} x^{k-1-j} =: p_{k-1}(x), \tag{4.1}$$

$$H_2 = -\frac{1}{k!} x^k + \sum_{j=2}^{k} \frac{c_j}{(k-j)!} x^{k-j} =: p_k(x). \tag{4.2}$$

Here and in the remaining of the paper the subindex of a polynomial denote its degree. For then a further derivation of equation (3.19) together (3.13) yield

$$\ddot{x} = \dot{H}_1 = H_2 H_3 = p_k(x) p_{k-1}(x) =: q_{k(k-1)}(x),$$

and by multiplying both sides by $\dot{x}$ one gets

$$(\dot{x})^2 = r_{k(k-1)+1}(x).$$

As a consequence, x can be explicitly integrated by inverting the hyperelliptic integral

$$\int \frac{dx}{\sqrt{r_{k(k-1)+1}(x)}} \tag{4.3}$$

4.2. **Optimal curves.** As projection of extremal curves, optimal solutions necessarily satisfy

$$\dot{x} = \sqrt{1 - p_k^2(x)}, \tag{4.4}$$

$$\dot{y} = p_k(x), \quad \text{and} \tag{4.5}$$

$$\dot{\theta}_j = \frac{x^j}{j!} p_k(x), \quad \text{for} \quad j = 1, \dots k \tag{4.6}$$

5. Motion planning for Goursat structures.

We use now the geometric information provided by the Pontryagin maximum principle and the hyperelliptic curve described above, for deriving the geometric features that optimal trajectories in the plane $\{(x, y)\} \simeq \{(x, y, 0, \dots, 0)\}$ necessarily have.

To begin with, we consider the three-dimensional space with coordinates

$$(x, u_1, z) = (x, \dot{x}, \dot{y}) = (x, u_1, u_2) = (H_{n-1}, H_1, H_2),$$

and the following two cylinders:

$$\mathcal{C}_1 = \{(x, u_1, z) \mid u_1^2 + z^2 = 1\} \quad \text{and} \quad \mathcal{C}_2 = \{(x, u_1, z) \mid z = p_k(x)\}.$$

We can assume that the $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$ and by choosing the initial conditions properly, we can assume that this intersection is a smooth, connected, closed and simple curve. We denote by $\mathcal{C}$ the parametrized curve that is the projection of $\mathcal{C}_1 \cap \mathcal{C}_2$ to the plane $\{(x, u_1)\} \simeq \{(x, u_1, 0)\}$, and call it the *cacahuète curve* of the problem, taking equation (4.4) into consideration we have:

$$\mathcal{C} = \left\{ (x, u_1) \mid u_1 = \sqrt{1 - p_k^2(x)} \right\}.$$

The cacahuète is a smooth, closed and simple curve that has both, vertical and horizontal symmetries, and by choosing the initial conditions properly it can be assumed that it is centered at the origin.

One can see on figure (5) how the cacahuète comes.

The cacahuète encodes the information of extremal curves in the plane $\{(x, y)\} \simeq \{(x, y, 0, \dots, 0)\} \subset \{(x, y, \theta_1, \dots, \theta_k)\}$. Let $\mathcal{E}$ be such an arc-length parametrized extremal curve in this plane, taking equation (4.5) into account we have:

$$\mathcal{E} = \{(x, y) \mid \dot{y} = p_k(x)\}.$$

We assume $\mathcal{E}$ to be centered at the origin. Moreover, we want the curve $\mathcal{E}$ to have total length 1 (or ε), but this can be obtained a posteriori by homogeneity.

In order to depict the curve $\mathcal{E}$ we have the freedom of choosing the coefficients of the polynomial $p_k(x)$ and the initial condition $y(0)$. Also we have to take into consideration the following *interpolation conditions* (which are independent of any translation of coordinates):

(1) The coordinate y is periodic, $y(1) = y(0)$.
(2) The *moments*

$$m_i = \int_{\mathcal{E}} x^i dy$$

are all zero for $i = 1, \dots, n-3$, which corresponds to the fact that the coordinates $\theta_1, \dots, \theta_{n-3}$, given by (4.5) are all periodic. Observe that the first moment m_1 is the area swept out by the curve, whereas the last moment m_{n-2} is not only non vanishing but also the one to be maximized.

The curves $\mathcal{E}$ are symmetric with respect to the x-axis but we do not know if they are symmetric with respect to the y-axis, however we make this assumption, a priori reasonable.

Remark 5.
- *Since we will be able to find geodesics meeting this assumption plus the boundary conditions, it is reasonable to expect that these geodesics are optimal.*
- *It is clear the union set of our centered solutions, is symmetric with respect to the y-axis, but it is not clear that the minimal solution is unique.*
- *Due to the interpolation requirements, it is clear that the projection of our curve on the plane (x, y) has to be periodic. Moreover, with the group law explicitly exhibited in (3.1) we can follow the same proof as in the papers* [8, 9]*, to show that this curve is in fact smooth-periodic.*

Under these symmetry considerations and depending on the parity of n, (which is the same parity of k) certain moments are automatically zero and the description of $\mathcal{E}$ can always be completed.

- If n is even, the odd moments are zero. The polynomial $p_k(x)$ has even degree and by the symmetry considerations, it has no terms of odd degree. Then, if we chose a monic polynomial, it remains $\frac{k}{2}$ free coefficients, that have to be used to vanish $\frac{k}{2} - 1$ moments (plus the zero-moment y).

  For instance, for $n = 6$, we have to chose only two coefficients to make y periodic and to vanish the moment m_2.
- if n is odd, the even moments are zero. The polynomial $p_k(x)$ has odd degree and by the symmetry considerations, it has no terms of even degree. Then, if we chose a monic polynomial, it remains $\frac{k-1}{2}$ free coefficients, that have to be used to vanish $\frac{k-1}{2}$ moments and the value $y(0)$ to make y periodic, which can be done as an independent (trivial) step.

  For instance, for $n = 7$, we have to chose two coefficients to vanish two moments, and we chose the value $y(0)$ to make y vanishing at a quarter period.

From this analysis we can conclude that, at the end we have as many free parameters (plus one that accounts for the initial condition $y(0)$) as the number of moments that have to vanish.

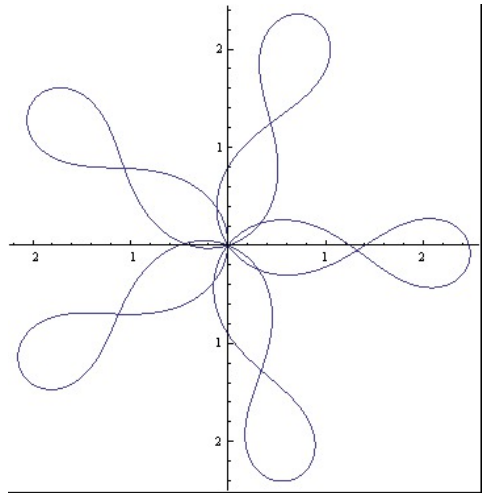

FIGURE 6. Other periodic extremals for order 3 (length-4) brackets

In practice, we use the only following numerical integration:

$$\frac{dy}{dx} = \frac{p_k(x)}{\sqrt{1 - p_k(x)^2}},$$

and compute the solution for a quarter period only. The x corresponding to a quarter period (denoted by $x_{\frac{1}{4}}$) is just given by the largest real root of $p_k(x)$. The value $y(0)$ follows. Solutions over the other 3 intervals are obtained by symmetry.

Hence, for both $n = 6, 7$, the computation reduces to this numerical integration (over a quarter period), and to performing shooting on 2 coefficients of $p_k(x)$. We obtained the numerical results that are presented in the introduction.

6. Conclusions and some open questions

In this paper we propose to use our entropy method of previous papers to treat the motion-planning problem for Goursat sub-riemannian metrics in arbitrary dimensions. We put in evidence a class of control trajectories and state trajectories that are in a sense universal and optimal, and therefore we think that these signals are the natural ones for dealing with motion planning questions. This conclusion is reinforced by the fact that it fits with the conclusions (in the generic cases, but for low order brackets) of our previous studies.

In particular, our results apply to the motion planning of the car with trailers, which is the prototype of a Goursat structure. In this case, moreover, the method can be used in a direct way (not asymptotic) to interpolate non-admissible paths by means of admissible ones.

In our view, there are many interesting issues to address in this regard, but we just want to bring the attention to the following two challenging questions:

- Find arguments (like it was done in the paper [9]) to prove that our (symmetric) Goursat extremals are actually optimal.
- Prove (or find counterexample) that in the **generic** case the geometric pattern of the pictures of optimal extremals persists, and depends only on

the length of the Lie brackets of the highest order. This is not so clear: when the order increases, the number of interpolation conditions increases much more (contrarily to what happens in the Goursat case), and it might happen that some more complicated periodic extremals come into the picture. Some of these more complicated extremals were exhausted in [9]. We show one of them in the figure 6.

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