A.A. Agrachev and J.-P. Gauthier, **On the subanalyticity of Carnot-Carathéodory distances**. *Annales de l'Institut Henri Poincaré; Analyse non-linéaire* **18**, No. 3, 2001, pp. 359 -382. (Item No. 1 831 660)

FEATURED REVIEW

This paper proves a number of spectacularly strong results on subanalyticity properties of Carnot-Carathéodory distances arising from generic real analytic subriemannian metrics. The main theorems say that: (T1) *generically, the germ at a point* q_0 *of the function* $q \mapsto \rho(q) \stackrel{\text{def}}{=} \text{dist}(q, q_0)$ *is subanalytic if the dimension* n *of the manifold and the dimension* k *of the distribution satisfy* $n \leq (k-1)k+1$, (T2) *generically (and, in fact, on the complement of a set of distributions of infinite codimension), small balls* $\{q : \rho(q) \leq r\}$ *are subanalytic if* $k \geq 3$, and (T3) *generically, the germ of* ρ *at* q_0 *is not subanalytic if* $n \geq (k-1)\Big(\frac{k^2}{3} + \frac{5k}{6} + 1\Big)$. Furthermore, several of the paper's results are much more concrete than the word "generically" might suggest, since they are proved by exhibiting a very explicit set of conditions on the distribution—which happen to be generic—under which the conclusion holds. For example, (T1) is a consequence of the much stronger result asserting that (T4) *for every real analytic subriemannian structure whose distribution is "medium fat,"* ρ *is subanalytic at* q_0, together with the observation (proved in [1]) that medium fatness is a generic property if $n \leq (k-1)k+1$. Statement T4 is a far-reaching generalization of earlier work on the subject, where subanalyticity was proved for "strongly bracket-generating" (also known as "fat") distributions, a very restrictive class introduced by R. Strichartz, cf. [5, 6].

These results continue the work by several authors (e.g., Sussmann [7], Ge [3], Agrachev-Sarychev [1], Jacquet [4]) who had investigated the subanalyticity of Carnot-Carathéodory distances and proved similar conclusions for much more restrictive classes of metrics. In a broader sense, the paper belongs to a program of research espoused by some differential-geometric control theorists, who proposed to study properties of optimal controls going beyond the first-order conditions given by the Pontryagin Maximum Principle, with the objective of establishing compactness results that might lead to conclusions about the structure of the value function. What makes this paper a truly remarkable tour de force is how far-reaching the new results are, and how the proofs combine a rich variety of techniques in ingenious and often unexpected ways.

I will now state the main definitions and results more precisely, and then outline the main ideas of some the proofs.

If $\nu = +\infty$ or $\nu = \omega$, a *distribution of class* C^ν on a manifold M of class C^ν is a linear subbundle Δ of the tangent bundle TM of M. The *dimension*—or *rank*— of Δ is its fiber dimension. A distribution Δ is *bracket-generating* if the Lie algebra $L(\Delta)$ of vector fields generated by the set $\Gamma^\infty(\Delta)$ of all sections of Δ of class C^∞ satisfies the "Hörmander condition" that $\{V(q) : V \in L(\Delta)\} = T_qM$ at every $q \in M$, where T_qM is the tangent space of M at q. A *subriemannian manifold of class* C^ν is a triple $\mathcal{M} = (M, \Delta, g)$, where M is a manifold of class C^ν, Δ is a bracket-generating distribution of class C^ν on M, and g is a Riemannian metric on Δ (i.e., a map $q \mapsto g_q$ such that g_q is a symmetric positive definite bilinear form on $\Delta(q)$ for each q) of class C^ν. An *admissible path* is a Lipschitz integral arc of Δ. Clearly, an admissible path $\gamma : [a, b] \mapsto M$ has a well-defined *length* $\|\gamma\|$, given by $\|\gamma\| \stackrel{\text{def}}{=} \int_a^b \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt$. If M is connected, then any two points q, q' can be joined by an admissble path γ, so we can define the *distance* $d_{\mathcal{M}}(q, q')$ to be the infimum of the lengths of all admissible paths from q to q' Given q, if q' is sufficiently close to q, then the infimum defining $d_{\mathcal{M}}(q,q')$ is in fact a minimum.

A distance function $d_{\mathcal{M}}$ arising from a subriemannian manifold $\mathcal{M} = (M, \Delta, g)$ of class C^ν with M connected is a *Carnot-Carathéodory* distance of class C^ν.

Finally, if M is a real-analytic manifold, then a subriemannian structure on M with a k-dimensional distribution is a section of class C^∞ of the bundle $SBR^k(M)$ over M whose fiber $SBR^k_q(M)$ at each point q of M is the set of pairs (S,g) consisting of a k-dimensional subspace of the tangent space T_q and a positive definite quadratic form g on S. The set $\mathrm{Sec}^\infty(SBR^k(M))$ of all such sections can be endowed with the C^∞ Whitney topology. A property of real analytic subriemannian structures holds *generically* if there is an open dense subset $\mathcal{G}$ of $\mathrm{Sec}^\infty(SBR^k(M))$ such that the property holds for all real analytic members of $\mathcal{G}$.

I now move on to the proofs. To convey the flavor, and show the rich and varied ideas and techniques that are brought together by the authors, I shall limit myself to one of the results, namely, (T2).

Since the results are local, we may take M to be $\mathbb{R}^n$ and work on a neighborhood $\mathcal{N}$ of q_0 on which there is a g-orthonormal basis $\mathbf{X} = (X_1, \ldots, X_k)$ of C^ω sections of the distribution Δ. The admissible curves are then exactly the Lipschitz curves $\xi : I_\xi \mapsto \mathcal{N}$ defined on some compact interval $I_\xi = [a_\xi, b_\xi]$ that satisfy $\dot\xi(t) = \sum_{i=1}^k u^i(t) X_i(\xi(t))$ for a.e. $t \in I_\xi$ for some "control" $\mathbf{u}_\xi = (u^1_\xi, \ldots, u^k_\xi) \in L^\infty(I_\xi, \mathbb{R}^k)$, which is then uniquely determined by ξ. Clearly, if $\xi : [a,b] \mapsto \mathcal{N}$ is admissible, then the subriemannian length $\Lambda_{\mathcal{M}}(\xi)$ of ξ is the L^1 norm of $\mathbf{u}_\xi$. Furthermore, ξ can be reparametrized so as to yield an admissible curve $\gamma : [0,1] \mapsto \mathcal{N}$ which is PCAL (parametrized by constant times arc-length), i.e., having the property that $\|\mathbf{u}_\gamma(t)\| = \Lambda_{\mathcal{M}}(\xi) = \Lambda_{\mathcal{M}}(\gamma)$ for a.e. $t \in [0,1]$. Then the L^2 norm of $\mathbf{u}_\gamma$ is also equal to $\Lambda_{\mathcal{M}}(\gamma)$. If $\gamma : [0,1] \mapsto \mathcal{N}$ is a PCAL length-minimizer, and $\zeta : [0,1] \mapsto \mathcal{N}$ is an arbitrary admissible curve with the same endpoints, then $\|\mathbf{u}_\gamma\|_{L^2} = \|\mathbf{u}_\gamma\|_{L^1} = \Lambda_{\mathcal{M}}(\gamma) \leq \Lambda_{\mathcal{M}}(\zeta) = \|\mathbf{u}_\zeta\|_{L^1} \leq \|\mathbf{u}_\zeta\|_{L^2}$, so γ is also an energy-minimizer, i.e., γ minimizes the L^2 norm of the control among all admissible curves $\zeta : [0,1] \mapsto \mathcal{N}$ with the same endpoints as γ. Conversely, if γ is an energy-minimizer, then γ must be PCAL (for otherwise its PCAL reparametrization η would satisfy $\|u_\eta\|_{L^2} = \|u_\eta\|_{L^1} = \|u_\gamma\|_{L^1} < \|u_\gamma\|_{L^2}$), and γ is a lenght-minimizer. (Indeed, let $\zeta : [0,1] \mapsto \mathcal{N}$ be admissible with the same endpoints as γ. Let η be its PCAL reparametrization. Then $\Lambda_{\mathcal{M}}(\gamma) = \|\mathbf{u}_\gamma\|_{L^1} = \|\mathbf{u}_\gamma\|_{L^2} \leq \|\mathbf{u}_\eta\|_{L^2} = \Lambda_{\mathcal{M}}(\eta) = \Lambda_{\mathcal{M}}(\zeta)$.) Let $\mathcal{Y} \stackrel{\mathrm{def}}{=} L^2([0,1], \mathbb{R}^k)$, and let Ω be the set of all $\mathbf{u} \in \mathcal{Y}$ such that the Cauchy problem $\dot q(t) = \sum_{i=1}^k u^i(t) X_i(q(t))$, $q(0) = q_0$, has a solution $\xi_{\mathbf{u}} : [0,1] \mapsto \mathcal{N}$ (which is obviously unique). Then Ω is open in $\mathcal{Y}$, and the map $\mathbf{u} \mapsto \xi_{\mathbf{u}}$ is continuous from (Ω, weak) to $C^0([0,1], \mathbb{R}^n)$. Define the "endpoint map" $f : \Omega \mapsto \mathcal{N}$ by $f(\mathbf{u}) = \xi_{\mathbf{u}}(1)$ for $\mathbf{u} \in \Omega$. For $R > 0$, let $B_R = \{\mathbf{u} \in \mathcal{Y} : \|\mathbf{u}\|_{L^2} \leq R\}$, $U_R = \{\mathbf{u} \in \mathcal{Y} : \|\mathbf{u}\|_{L^2} = R\}$, $B^{\mathcal{M}}_R \stackrel{\mathrm{def}}{=} \{q \in M : \rho(q) \leq R\}$, $S^{\mathcal{M}}_R \stackrel{\mathrm{def}}{=} \{q \in M : \rho(q) = R\}$. Fix R such that $B_R \subseteq \Omega$. Then $B^{\mathcal{M}}_R \subseteq \mathcal{N}$, and $B^{\mathcal{M}}_R$ is compact, because $B^{\mathcal{M}}_R = f(B_R)$, B_R is weakly compact, and f is continuous from (Ω, weak) to $\mathbb{R}^n$.

For $0 < r < R$, let Ω^{min}_r be the set of energy-minimizing controls of norm r, that is, the set of all $\mathbf{u} \in \Omega$ such that $\|\mathbf{u}\|_{L^2} = r$ and $\rho(f(\mathbf{u})) = r$. Then Ω^{min}_r is strongly compact and $S^{\mathcal{M}}_r = f(\Omega^{min}_r)$. (Indeed, if $\hat{\mathbf{u}} = \{\mathbf{u}_j\}_{j=1}^\infty$ is a sequence in B_R, then $\hat{\mathbf{u}}$ has a subsequence $\tilde{\mathbf{u}} = \{\mathbf{u}_{j(\ell)}\}_{\ell=1}^\infty$ that weakly converges to a control $\mathbf{u} \in B_R$, and then $f(\mathbf{u}_{j(\ell)}) \to f(\mathbf{u})$ as $\ell \to \infty$. If $0 < r < R$ and $q \in S^{\mathcal{M}}_r$, then we can pick the $\mathbf{u}_j$ such that $\|\mathbf{u}_j\|_{L^2} \leq r + 2^{-j}$ and $f(\mathbf{u}_j) = q$. It follows that $\|\mathbf{u}\|_{L^2} \leq r$ and $f(\mathbf{u}) = q$. But then $\|\mathbf{u}\|_{L^2} = r$, because $\rho(q) = r$, so $\mathbf{u} \in \Omega_r^{min}$, and then $q \in f(\Omega_r^{min})$, showing that $S_r^{\mathcal{M}} = f(\Omega_r^{min})$. If $\hat{\mathbf{u}}$ is a sequence in Ω_r^{min} then the weak limit $\mathbf{u}$ of the weakly convergent subsequence $\tilde{\mathbf{u}}$ will satisfy $\|\mathbf{u}\|_{L^2} = r$, because $\rho(f(\mathbf{u})) = \lim_{\ell\to\infty} \rho(f(\mathbf{u}_{j(\ell)})) = r$. So $\tilde{\mathbf{u}}$ converges strongly to $\mathbf{u}$, proving the compactness of Ω_r^{min}.)

Since $S_r^{\mathcal{M}} = f(\Omega_r^{min})$, the subanalyticity of $S_r^{\mathcal{M}}$ would follow if we could somehow replace Ω_r^{min} by a compact subanalytic subset of some finite-dimensional space $\mathbb{R}^s$. To do this, a more detailed analysis of the minimizing controls is needed. Let f_r denote the restriction of f to the sphere U_r. Then U_r is a C^∞ Hilbert submanifold of $\mathcal{Y}$ of codimension one, and f_r is a map of class C^∞, depending smoothly on r for $0 < r < R$ (i.e., the map $]0, R[\times U_1 \ni (r, \mathbf{u}) \mapsto f(r\mathbf{u})$ is smooth). Let $\mathrm{Crit}(f_r)$ denote the set of critical points of f_r. Then $\Omega_r^{min} \subseteq \mathrm{Crit}(f_r)$. (Reason: if $\mathbf{u} \in \Omega_r^{min}$ but $\mathbf{u} \notin \mathrm{Crit}(f_r)$, then we can let $\mathbf{u}_s(t) = s\mathbf{u}(st)$ for $0 < s \le 1$, $0 \le t \le 1$; then $\|\mathbf{u}_s\|_{L^2} = sr$, $f(\mathbf{u}_s) = \xi_{\mathbf{u}}(s)$, and $\mathbf{u}_s \to \mathbf{u}$ as $s \uparrow 1$; therefore $\mathbf{u}_s \notin \mathrm{Crit}(f_{sr})$ if $1 - s$ is small enough; if we pick such an s, then $f(\mathbf{u}_{s+h}) \in f(U_{sr})$ if $0 < h < 1 - s$ and h is small enough, so $\rho(f(\mathbf{u}_{s+h})) \le sr$ for such h, contradicting the fact that $\mathbf{u}_{s+h} \in \Omega_{(s+h)r}^{min}$, since $\mathbf{u} \in \Omega_r^{min}$.)

The inclusion $\Omega_r^{min} \subseteq \mathrm{Crit}(f_r)$, which is a special case of the "Pontryagin Maximum Principle" of optimal control theory, is the basic necessary condition for a control $\mathbf{u}$ to be an energy-minimizer. Controls that belong to $\cup_r \mathrm{Crit}(f_r)$ are called "extremal." For $\mathbf{u} \in \Omega\backslash\{0\}$, let $E_{\mathbf{u}} = \{\mathbf{v} \in \mathcal{Y} : \langle \mathbf{v}, \mathbf{u}\rangle = 0\}$, and write $A_{\mathbf{u}} = Df(\mathbf{u})(E_{\mathbf{u}})$, $B_{\mathbf{u}} = Df(\mathbf{u})(\mathcal{Y})$. Then a control $\mathbf{u} \in U_r$ is extremal iff $A_{\mathbf{u}} \ne \mathbb{R}^k$. This can happen in three mutually exclusive ways: (1) $A_{\mathbf{u}} \ne \mathbb{R}^k$ but $B_{\mathbf{u}} = \mathbb{R}^k$, (2) $A_{\mathbf{u}} = B_{\mathbf{u}} \ne \mathbb{R}^k$, (3) $A_{\mathbf{u}} \ne B_{\mathbf{u}} \ne \mathbb{R}^k$. If $\mathbf{u}$ satisfies (1) or (3) then $\mathbf{u}$ (or its corresponding trajectory $\xi_{\mathbf{u}}$) is a *normal extremal*. If $\mathbf{u}$ satisfies (2) or (3) then $\mathbf{u}$ is an *abnormal extremal*. (Thus if $\mathbf{u}$ satisfies (3) then it is both normal and abnormal. If $\mathbf{u}$ satisfies (1) then it is *strictly normal*, and if it satisfies (2) then it is *strictly abnormal*.) Clearly, a control $\mathbf{u}$ is extremal iff there exists a vector $\lambda \in \mathbb{R}^k\backslash\{0\}$ such that $\langle \lambda, Df(\mathbf{u})v\rangle = 0$ for all $v \in E_{\mathbf{u}}$. In that case, there exists a unique $\nu \in \mathbb{R}$ such that $\langle \lambda, Df(\mathbf{u})v\rangle = \nu\langle \mathbf{u}, v\rangle$ for all $v \in \mathcal{Y}$. Then $\mathbf{u}$ is normal if λ can be chosen so that $\nu \ne 0$, and abnormal if λ can be chosen so that $\nu = 0$.

The functional $\mathcal{Y} \ni v \mapsto \langle \lambda, Df(\mathbf{u})v\rangle$ can be represented as follows. Let $\mathbb{R}_k$ be the space of k-dimensional real row vectors. Let $\mu_\lambda : [0, 1] \mapsto \mathbb{R}_k$ be the unique solution μ of the "adjoint equation" $\dot{\mu}(t) = -\sum_{i=1}^k u_\xi^i(t)\mu(t) \cdot \frac{\partial X_i}{\partial q}(\xi(t))$ such that $\mu(1) = \lambda$. Then $\langle \lambda, Df(\mathbf{u})v\rangle = \int_0^1 (\sum_{i=1}^k v_i(t)\, \langle \mu_\lambda, X_i(\xi_{\mathbf{u}}(t))\rangle\, dt$ for all $v \in \mathcal{Y}$. So $\mathbf{u}$ *is extremal iff there exists an "extremal adjoint vector" (EAV) for* $\mathbf{u}$*, i.e., a nontrivial solution* μ *of the adjoint equation for which there is a real constant* ν *(the "abnormal multiplier" associated to* μ*) such that* $\langle \mu(t), X_i(\xi_{\mathbf{u}}(t))\rangle = \nu u^i(t)$ *for* $i = 1, \ldots, k$ *for a.e.* t. Furthermore, $\mathbf{u}$ *is normal (resp. abnormal) if* μ *can be chosen so that* $\nu \ne 0$ *(resp.* $\nu = 0$*)*.

Let $h_i : T^*\mathcal{N} \mapsto \mathbb{R}$ be the momentum functions of the vector fields X_i, given by $h_i(q, p) \stackrel{\mathrm{def}}{=} \langle p, X_i(q)\rangle$ for $q \in \mathcal{N}$, $p \in T_q^*\mathcal{N}$. Define $h = \frac{1}{2}\sum_{i=1}^k h_i^2$. If μ is an EAV for $\mathbf{u}$, with abnormal multiplier ν, then the derivative of h along the curve $t \mapsto \Xi(t) = (\xi_{\mathbf{u}}(t), \mu(t))$ in $T^*\mathcal{N}$ is the sum $\sigma(t) = \sum_{i=1}^k \sum_{j=1}^k h_i(\Xi(t))u_j(t)\langle \mu(t), [X_j, X_i](\xi(t))\rangle$. But $\sigma(t) \equiv 0$, because $\sigma(t) = \nu \sum_{i=1}^k \sum_{j=1}^k u_i(t)u_j(t)\langle \mu(t), [X_j, X_i](\xi(t))$. So h is constant along Ξ. If $\nu \ne 0$, then we can normalize μ so that $\nu = 1$, and then the constant value $c_{h,\Xi}$ of h along Ξ is just $\|\mathbf{u}(t)\|$. So $\xi_{\mathbf{u}}$ is PCAL, and it is easily verified that the curve Ξ is an integral curve of the Hamilton vector field $\vec{h}$ arising from h. It follows that *an admissible curve* $[0, 1] \ni t \mapsto \xi(t) \in \mathcal{N}$ *is a normal extremal iff it is the projection on* $\mathcal{N}$ *of a curve* $[0, 1] \ni t \mapsto \Xi(t) \in T^*\mathcal{N}$ *in the cotangent bundle of* $\mathcal{N}$ *which is an integral curve of* $\vec{h}$ *such that the constant value* $c_{h,\Xi}$ *of* h *along* Ξ *is nonzero, and in that case the control*

$\mathbf{u}_\xi = (u^1_\xi, \ldots, u^k_\xi)$ *is given by* $u^i_\xi(t) = h_i(\Xi(t))$, *so* $\|\mathbf{u}\|_{L^2} = \sqrt{2c_{h,\Xi}}$.

Now, if $0 < r < R$ we let $C(r) = \{p \in T^*_{q_0}\mathcal{N} : \sum_{i=1}^k h_i(q_0,p)^2 = r^2\}$. For each $p \in C(r)$ we let $\tilde{\Gamma}_p$ be the maximal integral curve of $\vec{h}$ such that $\Gamma_p(0) = (q_0, p)$. Then the domain of $\tilde{\Gamma}_p$ contains the interval $\left[0, \frac{R}{r}\right]$. Let Γ_p be the restriction of $\tilde{\Gamma}_p$ to $[0,1]$. Write $\tilde{\Gamma}_p = (\tilde{\gamma}_p, \tilde{\mu}_p)$, $\Gamma_p = (\gamma_p, \mu_p)$, so γ_p is a normal extremal for to a control $\mathbf{v}_p$, and μ_p is a field of covectors along γ_p. Let φ_r be the map $C(r) \ni p \mapsto f(\mathbf{v}_p)$. Let $C_{min}(r) = \{p \in C(r) : \rho(\gamma_p(1)) = r\}$. Then $S^{\mathcal{M}}_r = \varphi_r(C_{min}(r))$. If it was true that (I) *every energy-minimizer is a normal extremal*, and (II) *if* $0 < r < R$, *then* $C_{min}(r)$ *contains a bounded subset* $C_b(r)$ *such that* $S^{\mathcal{M}}_r = \varphi_r(C_b(r))$, then the subanalyticity of $S^{\mathcal{M}}_r$ would follow. (Indeed, fix δ such that $\delta > 0$ and $B \stackrel{\text{def}}{=} \{q : \|q - q_0\| \le \delta\} \subseteq B^{\mathcal{M}}_{\frac{r}{2}}$. Let r' be such that $0 < r' < r$ and $B^{\mathcal{M}}_{r'} \subseteq B$. Pick compact subanalytic subsets K, K' of $C(r)$, $C(r')$, respectively, such that $C_b(r) \subseteq K$ and $C_b(r') \subseteq K'$. Let $J_1 = \{p \in K : \gamma_p(1) \in B\}$, $J_2 = \{p \in K : (\exists p')(\exists t)(p' \in K' \wedge 0 < t < \frac{r}{r'} \wedge \gamma_p(1) = \tilde{\gamma}_{p'}(t))\}$. Then $K \backslash (J_1 \cup J_2)$ is a subanalytic subset of K, so $\varphi_r(K \backslash (J_1 \cup J_2))$ is subanalytic, and $\varphi_r(K \backslash (J_1 \cup J_2)) = S^{\mathcal{M}}_r$, so $S^{\mathcal{M}}_r$ is subanalytic as well.)

Conditions (I) and (II) are obviously true if $k = n$, i.e., in the Riemannian case, but they are false in general. Furthermore, to prove subanalyticity of small subriemannian spheres it suffices to prove (I) and (II). The way Agrachev and Gauthier do it is a remarkable technical achievement, which elegantly brings combines a variety of methods from optimal control theory that had been developed for totally different purposes.

To begin with, one needs some results about the second variation of an extremal control, especially the theory of the "index." If F is a map of class C^2 from a Banach manifold $\mathcal{U}$ to a finite-dimensional manifold M, and $\bar{u}$ is a critical point of F, then F has a well defined *Hessian* (or "intrinsic second derivative") $\text{Hes}_{\bar{u}}F$ at $\bar{u}$. By definition, $\text{Hes}_{\bar{u}}F$ is a quadratic map from $\ker DF(\bar{u})$ to $\text{coker}\, DF(\bar{u})$, given by

$$\text{Hes}_{\bar{u}}F(v) = 2 \lim_{t \downarrow 0} t^{-1} \Theta\Big(F(\gamma(\sqrt{t})) - F(\bar{u})\Big) \quad \text{for} \quad v \in \ker DF(\bar{u}),$$

if $\Theta : T_{F(\bar{u})}M \mapsto \text{coker}\, DF(\bar{u})$ is the canonical projection, and $\gamma : [-\varepsilon, \varepsilon] \mapsto \mathcal{U}$ is any C^2 curve such that $\gamma(0) = \bar{u}$ and $\dot{\gamma}(0) = v$. If $\lambda : T_{F(\bar{u})}M \mapsto \mathbb{R}$ is a nontrivial linear functional that annihilates $\text{im}\, DF(\bar{u})$, then we can define the *index* $\text{ind}_F(\bar{u}, \lambda)$ to be the supremum of the numbers $\dim V - \dim \text{coker}\, DF(\bar{u})$, taken over all linear subspaces V of $\ker DF(\bar{u})$ such that the quadratic form $\lambda \cdot \text{Hes}_{\bar{u}}F$ is positive definite on V. Then $-\dim M \le \text{ind}_F(\bar{u}, \lambda) \le +\infty$. (If ξ is a PCAL extremal of length r, and λ, ν are a covector in $T^*_{\xi(1)}M \backslash \{0\}$ and a real number such that $\langle \lambda, Df_r(\mathbf{u}_\xi)v \rangle = \nu \langle \mathbf{u}_\xi, v \rangle$ for all $v \in \mathcal{Y}$, then $(\lambda \cdot \text{Hes}_{\mathbf{u}_\xi} f_r)(v) = \langle \lambda, D^2 f(\mathbf{u}_\xi)(v) \rangle - \nu \|v\|^2$ for all $v \in \mathcal{Y}$.)

The index $\text{ind}(\xi, \lambda, \nu) \stackrel{\text{def}}{=} \text{ind}_{f_r}(\mathbf{u}_\xi, \lambda)$ of the extremal ξ, relative to the multipliers λ, ν, is then well defined. The key facts about the index are then (F1) *the function* $(\mathbf{u}, \lambda, \nu) \mapsto \text{ind}(\xi_{\mathbf{u}}, \lambda, \nu)$ *is lower semicontinuous*, and (F2) *if* $\mathbf{u}$ *is a minimizer then* $\text{ind}(\xi_{\mathbf{u}}, \lambda, \nu) < 0$ *for some* (λ, ν). Furthermore, there is a class of abnormal extremals, called "Goh extremals," with important special properties. Precisely, a *Goh multiplier* for an extremal ξ is a multiplier $(\lambda, 0)$ such that the adjoint vector $\mu_\lambda(t)$ annihilates the vectors $[X_i, X_j](\xi(t))$ for all i, j, t. It can then be proved that (F3) *if* $\text{ind}(\xi_{\mathbf{u}}, \lambda, 0) < 0$ *then* $(\lambda, 0)$ *is a Goh multiplier for* ξ. A *Goh extremal* is an extremal that admits a Goh multiplier. Then *if there do not exist Goh extremals it follows that the spheres* $S^{\mathcal{M}}_r$ *are subanalytic for small* r. (The argument is as follows. First, every minimizer must be a normal extremal, because if a minimizer ξ was strictly abnormal then (F2) tells us that ξ admits a multiplier (λ, ν) such that $\text{ind}(\xi, \lambda, \nu) < 0$, and then the strict abnormality of ξ implies that $\nu = 0$, so $(\lambda, 0)$ must be a Goh multiplier for ξ, and we have reached a contradiction. Therefore (I) holds. To prove (II), we define $C_b(r)$ to be the set of all $p \in C_{min}(r)$ such that $\text{ind}(\gamma_p, \mu_p(1), \nu) < 0$ for some ν. Then $\varphi_r(C_b(r)) = S_r^{\mathcal{M}}$ thanks to (F2). Let us prove that $C_b(r)$ is compact. Pick a sequence $\{p_j\}_{j=1}^{\infty}$ in $C_b(r)$. Then the controls $\mathbf{v}_{p_j}$ belong to Ω_r^{min}, which is compact. Therefore, by passing to a subsequence, we may assume that $\mathbf{v}_{p_j}$ converge to a limit $\mathbf{v}$, which gives rise to a minimizing trajectory ξ. If the sequence $\{p_j\}_{j=1}^{\infty}$ is unbounded, then we may assume, after passing to a subsequence, that $\|p_j\| \to \infty$. If we let $\tilde{p}_j = \frac{p_j}{\|p_j\|}$, then we may assume, after passing to a subsequence, that the $\tilde{p}_j$ converge to a limit $\tilde{p}$. Since each $\mu_{\tilde{p}_j}$ is an adjoint vector for ξ_j with multiplier $\frac{1}{\|p_j\|}$, the function $\mu_{\tilde{p}}$ is an adjoint vector for ξ with multiplier 0, i.e., an abnormal multiplier. The lower semicontinuity of the index implies that $\text{ind}(\xi, \mu_p(1), 0) < 0$. But then ξ is a Goh extremal, and we have reached a contradiction. So $\{p_j\}_{j=1}^{\infty}$ is bounded, and then it has a subsequence that converges to a $p \in C(r)$. The corresponding control $\mathbf{v}_p$ is then $\mathbf{v}$, which is a minimizer, so $p \in C_{min}(r)$. Finally, the lower semicontinuity of the index implies that $p \in C_b(r)$.)

To conclude the proof of (T2), I shall now sketch the beautiful argument used by Agrachev and Gauthier to show that for $k \geq 3$, generically, there are no Goh extremals. Suppose ξ is a Goh extremal, so that there exists an adjoint vector $t \mapsto \mu(t)$ satisfying the conditions $\langle \mu(t), X_i(\xi(t)) \rangle = 0$ and $\langle \mu(t), [X_i, X_j](\xi(t)) \rangle = 0$ for all t, i, j. Assume, in addition, that $\mathbf{u}_\xi \in C^\infty$. Differentiation of the $\frac{k(k-1)}{2}$ identities $\langle \mu(t), [X_i, X_j](\xi(t)) \rangle = 0$, for $i < j$, yields $\sum_{\ell=1}^k u^\ell(t) \langle \mu(t), [X_\ell, [X_i, X_j]](\xi(t)) \rangle = 0$. This is a system of $\frac{k(k-1)}{2}$ equations in the k unknowns $u^1(t), \ldots, u^k(t)$. If $k \geq 3$ then $\frac{k(k-1)}{2} \geq k$, so the system contains at least k equations, and then the existence of a nontrivial solution requires at least one condition involving the coefficients. Successive differentiations yield more conditions involving Lie brackets of higher and higher orders. The final result, after one carefully keeps track of all these identitites (as the authors do in the paper), is that if $k \geq 3$ then the conditions for existence of smooth Goh extremals are only satisfied on a set of infinite codimension. So, generically, no smooth Goh extremals can exist if $k \geq 3$.

Finally, one has to infer the nonexistence of Goh extremals from the nonexistence of *smooth* Goh extremals. At this point, the following remarkable result from control theory comes to the rescue: *for a control system $\dot{q} = f(q, u)$, $u \in U$, with U compact subanalytic and f real analytic in q and u, if two points $\bar{q}$, $\hat{q}$ are such that there exists a trajectory $[a, b] \ni t \mapsto q(t)$ for a measurable control $[a, b] \ni t \mapsto u(t) \in U$ such that $q(a) = \bar{q}$ and $q(b) = \hat{q}$, then there exists a trajectory $[a, b] \ni t \mapsto \tilde{q}(t)$ such that $\tilde{q}(\tilde{a}) = \bar{q}$ and $\tilde{q}(\tilde{b}) = \hat{q}$, for a control $[\tilde{a}, \tilde{b}] \ni t \mapsto \tilde{u}(t) \in U$ which is analytic on an open dense subset of* $[\tilde{a}, \tilde{b}]$. This rather surprising fact was proved in 1986 in Sussmann [8] (cf. also [9] for a more detailed proof) as part of a study of regularity properties of optimal controls, and also by Gauthier and Kupka in 1996 in [2] for different reasons, arising from their work on observability. In our situation, it is used to prove that *if there exist no smooth nontrivial Goh controls then there exist no nontrivial Goh controls.* This is done by considering the system consisting of the equations $\dot{q} = \sum_{i=1}^k u^i X_i(q)$, $\dot{\mu} = -\sum_{i=1}^k u^i \mu \cdot \frac{\partial X_i}{\partial q}(q)$, $\dot{z} = \sum_{i=1}^k \langle \mu, X_i(q) \rangle^2 + \sum_{i=1}^k \sum_{j=1}^k \langle \mu, [X_i, X_j](q) \rangle^2$, for q, μ, and a new scalar variable z. If $\xi : [a, b] \mapsto \mathcal{N}$ is a Goh extremal, then ξ gives rise, in an obvious way, to a solution Ξ of the new system, going from a point $(\xi(a), \bar{\mu}, 0)$ to a point $(\xi(b), \hat{\mu}, 0)$. It follows that there is a trajectory $\tilde{\Xi}$ going from $(\xi(a), \bar{\mu}, 0)$ to $(\xi(b), \hat{\mu}, 0)$, for a control $\tilde{u}$ which is analytic on an open dense subset of its domain. Clearly, $z \equiv 0$ along $\tilde{\Xi}$. If I is a nontrivial interval on which $\tilde{u}$ is analytic, then the restriction of $\tilde{\Xi}$ to I is a nontrivial smooth Goh extremal, and the proof is complete.

This concludes the outline of the proof of (T2). The proofs of the other two main results require other ideas, in particular a careful analysis of nilpotent approximations of subriemannian structures. Lack of space prevents me from going into the details and doing full justice to the technical virtuosity of the authors, but the proof that has just been sketched for (T2) should suffice to establish that this is a major, impressive, masterly piece of work, which represents a significant step forward in our understanding of subriemannian metrics, and constitutes a magnificent success story for the field of differential-geometric control theory.

Reviewed by *H. J. Sussmann*[1]

References

[1] Agrachev, A.A., and A.V. Sarychev, "Sub-Riemannian metrics: minimality of abnormal geodesics versus subanalyticity." *J. ESAIM: Control, Optimization and Calculus of Variations* **4**, 1999, pp. 377-403.

[2] Gauthier, J.-P., and I. Kupka, "Observability for systems with more outputs than inputs and asymptotic observers." *Math. Zeitschr.* **223**, 1996, pp. 47-78.

[3] Ge Zhong, "Horizontal path spaces and Carnot-Carathéodory metric." *Pacific J. Math* **161**, 1993, pp. 255-286.

[4] Jacquet, S., "Subanalyticity of the sub-Riemannian distance." *J. Dynam. Control Systems* **5**, no. 3, 1999, pp. 303–328.

[5] Strichartz, R., "Sub-Riemannian Geometry." *J. Diff. Geometry* **24**, 1983, pp. 221-263.

[6] Strichartz, R., "Corrections to 'Sub-Riemannian Geometry'." *J. Diff. Geometry* **30**, 1989, pp. 595-596.

[7] Sussmann, H.J., "Optimal control and piecewise analyticity of the distance function." In *Optimization and Nonlinear Analysis (Haifa, 1990)*, A. Ioffe & S. Reich Eds., Pitman Res. Notes Math. **244**, Longman Sci. Tech., Harlow, 1992, pp. 298-310.

[8] Sussmann, H.J., "Trajectory regularity and real analyticity: some recent results." In *Proc. 25th IEEE Conference on Decision and Control (Athens, Greece, Dec. 1986)*, IEEE publications, New York, 1986, pp. 592-595.

[9] Sussmann, H.J., "Some optimal control applications of real analytic stratifications and desingularization." In *Singularities Symposium—Lojasiewiz 70*, B. Jakubczyk, W. Pawlucki and J. Stasica Eds., Banach Center Publications Vol. 44, Polish Academy of Sciences, Warsaw, Poland, 1998, pp. 211-232.

[1]Department of Mathematics; Rutgers University; Piscataway, NJ 08854-8019, USA. E-mail: *sussmann@math.rutgers.edu*. Web site: *http://www.math.rutgers.edu/~sussmann*.

Supported in part by NSF Grant DMS98-03411-00798.