EIGENSTRUCTURE ASSIGNMENT FOR STATE-CONSTRAINED LINEAR CONTINUOUS TIME SYSTEMS

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Abstract:
Linear state constraints define a polyhedron in the state space of a dynamical system. If this polyhedron can be made positively invariant by state feedback, the constraints are satisfied all along the trajectory for any initial point in this polyhedron. In this paper, some algebraic invariance relations are established for continuous-time linear systems. The existence of a solution to the corresponding set of equations and inequalities depends on the structural properties of an associated input-output system. In particular, the number of invariant zeros of this system should be appropriate. If these conditions are satisfied, the problem can easily be solved by eigenvalue and eigenvector assignment.

Keywords:
Constraint theory, Eigenvalue assignment, Eigenvector assignment, Invariance, Linear Systems, Multivariable control systems, Stability, Zeros

INTRODUCTION

Most industrial systems are subject to state and control limitations. A major issue in modern control design is direct integration of these constraints in the regulation scheme. In many cases, this objective can be met by constructing positively invariant domains included in the set of constraints (see for instance Gutman and Hagander (1985)). This method directly applies to the case of state constraints. But it can also be used in the case of control constraints under state feedback regulation laws (Hennet and Beziat, 1990). A non-empty domain Ω is positively invariant for a dynamical system if for any initial state in Ω, the state trajectory remains in Ω.

Asymptotical stability of a linear system is equivalent to the existence of positively invariant domains (Kalman and Bertram, 1960). The shape of these domains depends on the selected Lyapunov functions. Generalized $L_{\infty}$ norms can be selected as non-quadratic Lyapunov functions. They generate positively invariant polyhedra well-fitted to linearly constrained control problems. Until now, most of the theoretical and applicative work in this area deals with the discret-time

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case. In particular, some necessary and sufficient conditions for a given polyhedron to be a positively invariant set of a linear discrete-time system were established by Bitsoris (1988a, 1988b), Benzaouia and Burgat (1988) and Hennet (1989). But recently, continuous-time positive invariance conditions for some convex polyhedral domains were presented and applied to the solution of the Linear Constrained Regulation Problem (LCRP) by Vassilaki and Bitsoris (1989). Under state-feedback regulation laws, these relations can be interpreted as eigenstructure assignment properties. This paper extends these results and interprets them structurally. The design problem is then re-formulated as an eigenstructure assignment problem, for which a solution exists under some geometric conditions.

In this paper, we study the case of continuous-time linear systems under symmetrical linear state constraints. And we formulate some structural properties which permit to easily find linear regulation laws guaranteeing positive invariance of the constrained set. This approach combines the generic conditions for the existence of positively invariant polyhedra with the specific conditions for assigning the closed-loop eigenvectors and eigenvalues at the required locations.

PROBLEM PRESENTATION AND PRELIMINARY RESULTS

The continuous-time linear dynamical system to be controlled is represented by the following state-equation:

\[ \dot{x}(t) = Ax(t) + Bu(t) \text{ for } t \geq 0 \]  

with \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( m \leq n \) and \( \text{rank}(B) = m \).

\( (A, B) \) is supposed to be controllable.

A set of symmetrical linear constraints applies to the state vector, \( x(t) \). They generate the symmetrical polyhedral domain \( S(G, \rho) \) defined by:

\[ S(G, \rho) = \{ x \in \mathbb{R}^n : -\rho \leq Gx \leq \rho \} \]  

with \( G \in \mathbb{R}^{s \times n} \), \( s \leq n \), \( \text{rank}(G) = s \), \( \rho \in \mathbb{R}^s \) and \( \rho_i > 0 \) for \( i = 1, \ldots, s \).

Under a closed-loop linear regulation law: \( u(t) = Fx(t) \), with \( F \in \mathbb{R}^{m \times n} \), the evolution of the system is described by:

\[ \dot{x}(t) = A_0x(t) \text{ with } A_0 = A + BF \]  

The gain matrix to be constructed should drive the state vector to zero while maintaining it in the symmetrical polyhedral domain \( S(G, \rho) \).

In a more general context, let \( R[Q, \omega] \) be a nonempty convex polyhedron of \( \mathbb{R}^n \). It can be characterized by a matrix \( Q \in \mathbb{R}^{r \times n} \) and a vector \( \omega \in \mathbb{R}^r \), with \( \omega > 0 \), and defined by:

\[ R[Q, \omega] = \{ x \in \mathbb{R}^n : Qx \leq \omega \} \]  

In the continuous-time framework, the trajectory characterization of the invariance property of \( R[Q, \omega] \) can be defined by:

\[ Qx_0 \leq \omega \implies Qe^{At}x_0 \leq \omega , \ \forall \ x_0 \in R[Q, \omega] , \ \forall t \geq 0. \]  

The following proposition provides necessary and sufficient algebraic conditions for the positive invariance of \( R[Q, \omega] \). It extends the result for polyhedra with \( \text{rank}(Q) = n \) stated in Vassilaki
and Bitsoris (1989). The invariance relations (6) and (7) can be easily obtained from some duality properties in Linear Programming, and more precisely from a repeated use of Farkas’ lemma. And no assumption on the rank of matrix $Q$ is required. A detailed proof of this result can be found in Castelan and Hennet (1991).

**Proposition 2.1**

The polyhedron $R[Q, \omega]$ is a positively invariant set of system (3) if and only if there exists a matrix $H \in \mathbb{R}^{rs}$, with $(H)_{ij} \geq 0$ for $i \neq j$, such that:

\[
\begin{align*}
HQ &= QA_0 \\
H\omega &\leq 0
\end{align*}
\]  

(6) \hspace{1cm} (7)

A specialized version of proposition 2.1 can be obtained for characterizing the positive invariance of symmetrical polyhedra.

**Proposition 2.2**

The convex symmetrical polyhedron $S(G, \rho)$ is positively invariant for system (3) if and only if there exists a matrix $K \in \mathbb{R}^{ss}$ and a scalar $s_0 > 0$ such that:

\[
\begin{align*}
(-s_0I_s + K)G &= GA_0 \\
(-s_0I_s + |K|)\rho &\leq 0
\end{align*}
\]  

(8) \hspace{1cm} (9)

By definition, $|K|$ is the matrix of the absolute values of the components of matrix $K$.

Let us rewrite the polyhedron $S(G, \rho)$ under the form:

\[
R[Q, \omega] = \{ x \in \mathbb{R}^n : Qx \leq \omega \}
\]

with:

\[
Q = \begin{bmatrix} G \\ -G \end{bmatrix} \quad \text{and} \quad \omega = \begin{bmatrix} \rho \\ \rho \end{bmatrix}.
\]

Positive invariance of $S(G, \rho)$ is equivalent to the existence of a matrix $H \in \mathbb{R}^{2ss^2}$, written

\[
H = -s_0I_{2s} + \begin{pmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{21} & \hat{K}_{22} \end{pmatrix}
\]

where $s_0$ is a positive number and $\hat{K}_{11}, \hat{K}_{12}, \hat{K}_{21}, \hat{K}_{22}$ are four matrices with non-negative components in $\mathbb{R}^{ss}$, and such that relations (6) and (7) are satisfied.

Whenever this existence condition is satisfied, we can always set $K = \hat{K}_{11} - \hat{K}_{12}$, and it is easy to see that conditions (6) and (7) are also satisfied when selecting, instead of $H$, the symmetrical matrix $T \in \mathbb{R}^{2ss^2}$, $(T)_{ij} \geq 0$ for $i \neq j$, such that:

\[
T = -s_0I_{2s} + \begin{pmatrix} K^+ & K^- \\ K^- & K^+ \end{pmatrix}, \quad \text{with} \quad \begin{cases} (K^+)_{ij} = \max((K)_{ij}, 0) \geq 0 & \forall i, j = 1, \ldots, s \\ (K^-)_{ij} = \max(-(K)_{ij}, 0) \geq 0 & \forall i, j = 1, \ldots, s \end{cases}
\]  

(10)

Then, equation (8) is obtained from $HG = GA_0$, with, by definition:

\[
H = -s_0I_q + K
\]
And we also have from $|K| = K^+ + K^- :$

$$(-s_0 I_s + |K|) \rho \leq 0$$

Similarly to the discrete-time case (Vassilaki, Hennet and Bitsoris, 1988), the algebraic conditions (8) and (9) can be directly used for constructing stabilizing regulators letting $S(G, \rho)$ invariant. In particular, $s_0$ and the components of candidate matrices $K^+, K^-$ and $F$ can be obtained as the optimal solution of the following linear program:

Minimize $s_0$

subject to $-s_0 G + (K^+ - K^-)G - GBF = GA$

and $(K^+ + K^-)\rho - s_0 \rho \leq 0$

with $s_0, K^+, K^- \geq 0$

But this L.P. may have no feasible solution.

**STRUCTURAL CONDITIONS FOR POSITIVE INVARIANCE**

Conditions of proposition 2.2 cannot be satisfied by any system (3) and for any domain $S(G, \rho)$. The main results of this paper derive from the interpretation of relations (8) and (9), in terms of $(A,B)$-invariance and stability. When the structural properties of the triplet $(A,B,G)$ are appropriate, the invariance problem can be translated into an eigenstructure assignment problem.

**A - (A,B)-Invariance of KerG**

A geometric interpretation of relation (8) is presented in the following lemma.

**Lemma 3.1**

A necessary and sufficient condition for the existence of a couple of matrices $(H \in \mathbb{R}^{s \times s}, F \in \mathbb{R}^{m \times n})$ satisfying:

$$HG = G(A + BF)$$

is that Ker $G$ is an $(A,B)$-invariant subspace of system (3).

**Proof**:

(a) necessity

Under condition (11), if $Gx = 0$, then $G(A + BF)x = 0$; and this means $\dot{x} = A_0 x \in \text{Ker} G$ for all $x \in \text{Ker} G$.

Then, $Ax = \dot{x} - BFx$ implies $A\text{Ker} G \subseteq \text{Ker} G + \mathcal{B}$ where, by definition, $\mathcal{B}$ is the image (or range) of $B$.

Therefore, Ker $G$ is an $(A,B)$-invariant subspace of $\mathbb{R}^n$ and the gain matrix $F$ is ”a friend of Ker $G”$ (Wonham, 1985). In geometric notation,

$$(A + BF)\text{Ker} G \subseteq \text{Ker} G$$
(b) sufficiency
If \( \ker G \) is \((A,B)\)-invariant, then, there exists a matrix \( F \in \mathbb{R}^{m \times n} \) such that:
\[
Gx = 0 \implies G(A + BF)x = 0 \tag{13}
\]
Then,
\[
\ker G \subset \ker G(A + BF) \tag{14}
\]
From a well-known result of linear algebra, the orthogonal complementary subspace of \( \ker G \) in \( \mathbb{R}^n \) is spanned by the column-vectors of \( GT \) and the orthogonal complement of \( \ker G(A + BF) \) in \( \mathbb{R}^n \) is spanned by the column-vectors of \( [G(A + BF)]^T \).
Condition (14) implies:
\[
\text{Range}\{[G(A + BF)]^T\} \subset \text{Range}\{GT\} \tag{15}
\]
Clearly, relation (15) implies that each row-vectors of \( G(A + BF) \) can be written as a linear combination of the row-vectors of \( G \):
\[
[G(A + BF)]_i = [H_{i1} \ldots H_{is}]G \tag{16}
\]
By concatenation of conditions (16) for \( i = 1, \ldots, s \), we obtain:
\[
\exists H \in \mathbb{R}^{ss} ; G(A + BF) = HG
\]

Note that from any matrix \( H \in \mathbb{R}^{ss} \) and any positive scalar \( s_0 \) it is always possible to construct a matrix \( K = H + s_0I_s \). Consequently, condition (11) of lemma 3.1 is strictly equivalent to condition (8) of proposition 2.2.

Obviously, if \( \text{rank}(G) = n \ (\ker G = \{0\}) \), the condition of lemma 3.1 is always satisfied and the discussion is reduced to the analysis of the next subsection. Therefore, only the case \( \text{rank}(G) < n \) will now be considered.

If \( 0 < s < n \), geometric invariance of the subspace \( \ker G \) requires the existence of \((n - s)\) independent real generalized eigenvectors of \( A + BF \) in \( \ker G \), \((v_1, \ldots, v_{n-s})\), and of a matrix \( J_1 \in \mathbb{R}^{(n-s) \times (n-s)} \) having the real Jordan canonical form, satisfying:
\[
(A + BF)V_1 = V_1J_1 \quad \text{with} \quad V_1 = (v_1, \ldots, v_{n-s}). \tag{17}
\]
Consider the pole pencil of pair \((A,B)\), (Karcanias and Kouvaritakis 1978):
\[
S(\lambda) = [\lambda I - A \mid -B] \tag{18}
\]
where \( \lambda \) is a general complex frequency variable.
The transmission subspace of \( \lambda \), denoted \( Tr(\lambda) \), is defined by:
\[
Tr(\lambda) = \left\{ v \in \mathcal{C}^n : \exists \ w \in \mathcal{C}^m ; S(\lambda) \begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0 \right\} \tag{19}
\]
In any case, \( \ker S(\lambda) \) has dimension \( \geq m \). It is easy to show that \( Tr(\lambda) \) also has dimension \( \geq m \). Then, for any closed-loop eigenvalue \( \lambda_i \) of system (3), the corresponding assigned eigenvector, \( v_i \), belongs to \( Tr(\lambda_i) \). The associated direction in the input space satisfies:
\[
w_i = Fv_i \tag{20}
\]
Let $S(A, B, G)$ be defined by state equation (1) and by the output equation, with $y(t) \in \mathbb{R}^s$:

$$y(t) = Gx(t) \quad \text{for} \; t > 0$$  \tag{21}

For continuous-time linear systems, the system matrix (Rosenbrock, 1970), $P(\lambda)$, plays a crucial role in the study of the system frequency transmission (propagation and blocking) along some subspaces. And, it also enables to physically interpret some basic geometric concepts, such as $(A, B)$-invariance and controllability subspaces (Karcanas and Kouvaritakis, 1978). In a classical way, it is defined by:

$$P(\lambda) = \begin{bmatrix} \lambda I - A & -B \\ G & 0_{s \times m} \end{bmatrix}$$  \tag{22}

Then, the problem of satisfying the $(A, B)$-invariance condition on Ker $G$ reduces to the classical problem of zeroing the system outputs of $S(A, B, G)$ (Karcanas and Kouvaritakis, 1979), that is "to find complex frequencies, $\lambda_i = \mu_i + j\sigma_i$, and associated directions in the input and state spaces, respectively $w_i$ and $v_i$, which lead to null trajectories in the output-space".

The closed-loop eigenstructure in Ker $G$ is determined by these frequencies and associated directions. They should satisfy:

$$P(\lambda_i) \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$  \tag{23}

If $m < s < n$ and there is no structural degeneracy, equation (23) will not generally have any solution. And closed-loop invariance of Ker $G$ will not be obtained. We can therefore assume $s < n$ and $s \leq m$.

Let $d$ be the rank deficiency of matrix $GB$: rank($GB$) = $s - d$. To find the solutions of equation (23), define respectively right and left annihilators of matrices $B$ and $G$, $M \in \mathbb{R}^{n \times (n-s)}$ and $N \in \mathbb{R}^{(n-m) \times n}$, satisfying the following relations:

$$GM = 0_{s,n-s}$$  \tag{24}

$$NB = 0_{n-m,m}$$  \tag{25}

The degrees of freedom on the choice of matrices $N$ and $M$, and the property:

$$\text{rank}(NM) = n - m - d$$

allow to choose $N$ and $M$ so that (Kouvaritakis and Mac Farlane, 1976):

$$NM = \begin{bmatrix} I_{n-m-d} & 0_{n-m-d,m+d-s} \\ 0_{d,n-m-d} & 0_{d,m+d-s} \end{bmatrix}$$  \tag{26}

Any vector $v_i \in \mathbb{C}^n$ satisfying relation (23) belongs to Ker $G \subset \mathbb{C}^n$. Therefore, it is uniquely defined by the vector $z_i \in \mathbb{C}^{n-s}$ such that:

$$v_i = Mz_i$$  \tag{27}

Relation (23) can then be equivalently replaced by:

$$[\lambda_i I - A \mid -B] \begin{bmatrix} Mz_i \\ w_i \end{bmatrix} = 0$$  \tag{28}
The $m$ components of $w_i$ can be eliminated by left multiplication of (28) by matrix $N$, yielding:

$$\lambda_iNM - NAM \cdot z_i = 0 \quad (29)$$

Equations (23) and (29) have the same solutions $\lambda_i \in \mathbb{C}$. The polynomial matrix $[\lambda NM - NAM]$ is called the zero pencil (Kouvaritakis and Mac Farlane, 1976). It completely characterizes the finite zeros and the associated zero-directions of $S(A, B, G)$.

Using for matrix $NAM$ the same partitioning as for $NM$, we can write the zero pencil as follows:

$$\lambda NM - NAM = \begin{bmatrix} \lambda I_{n-m-d} - (NAM)_1 & -(NAM)_2 \\ -(NAM)_3 & -(NAM)_4 \end{bmatrix} \quad (30)$$

Under the assumptions $\text{rank}(G) = s$, $\text{rank}(B) = m$ and $s \leq m \leq n$, two types of solutions for (23) can be distinguished:

(a) The "controllable" solutions, which exist only if $s < m$ or $s = m$ with a singular pencil. They allow to construct a pair of zero-directions, $(v_i, w_i)$, for any value of complex frequency $\lambda_i$. The maximal number of independent state zero-directions associated to the selected frequencies generate the maximal controllability subspace in $\text{Ker} \ G$.

(b) The invariant zeros and associated invariant zero-directions: The invariant zeros are defined as the set of complex frequencies $\lambda_i$ which make $P(\lambda_i)$ rank deficient (see e.g. Shaked and Karcanias (1976)), that is such that $\text{rank}(P(\lambda_i)) < \min(n + s, n + m)$. The associated vectors $v_i$ and $w_i$ are respectively called invariant state zero-direction and invariant input zero-direction.

The rank deficiency of $P(\lambda_i)$, $d_i$, is the geometric multiplicity of $\lambda_i$. The algebraic multiplicity of $\lambda_i$ is $m_i = \sum_{j=1}^{d_i} m_{ij}$. It is the power of the invariant factor $(\lambda - \lambda_i)$ in the zero polynomial of $S(A, B, G)$.

If $m_i$ is strictly greater than $d_i$, the invariant zero-directions and generalized invariant zero-directions associated to the string $ij$ such that $m_{ij} > 1$ satisfy the sequence of equations (Mac Farlane and Karcanias, 1976):

$$P(\lambda_i) \begin{bmatrix} v'_{ij1} \\ w'_{ij1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

And, for $k = 2, ..., m_{ij}$,

$$P(\lambda_i) \begin{bmatrix} v'_{ijk} \\ w'_{ijk} \end{bmatrix} = \begin{bmatrix} -v'_{ij(k-1)} \\ 0 \end{bmatrix}$$

From theorem 4.2 in Karcanias and Kouvaritakis (1979), it is established that these invariant state zero-directions generate the maximal $(A,B)$-invariant subspace which has no intersection with $B$.

For clarifying the discussion, we can now separate the cases of regular and of singular pencils.

Case of regular zero pencils: If the zero pencil is square ($s = m$), and its determinant not always equal to 0 for any value of $\lambda$, it is said to be regular and its kernel is uniquely determined by the set of elementary divisors of $P(\lambda)$. In this regular case, the condition of $(A,B)$-invariance of $\text{Ker} \ G$ can be stated as follows:

**Proposition 3.2**
If the zero pencil is regular, a necessary and sufficient condition to span $\text{Ker } G$ with $n - m$ independent generalized state zero-directions is $\text{rank}(GB) = m$ (matrix $GB$ full-rank). Under this condition, it is always possible to obtain the $(A,B)$-invariance of $\text{Ker } G$ by assigning $n - m$ closed-loop poles at the zeros (with the same multiplicity). But a necessary condition for also obtaining the stability of the closed-loop system is that all the invariant zeros of $S(A,B,G)$ are stable (located in the left half-side of the complex plane).

Proof:
(a) case $d = 0$.
If matrix $GB$ is full-rank, that is for $d = 0$, then, from (26), $NM = I_{n-m}$. The $n - m$ invariant zeros of $S(A,B,C)$ are finite. They are the eigenvalues of matrix $NAM$. To each finite invariant zero, $\lambda_i$, with algebraic multiplicity $m_i$, corresponds an $(A,B)$-invariant subspace in $\text{Ker } G$, with dimension $m_i$. The direct sum of these elementary $(A,B)$-invariant subspaces is $\text{Ker } G$ itself.
System zeros are invariant under any state feedback (see e.g. Kouvaritakis and Mac Farlane (1976)). If any of these zeros is located outside the stable region of the complex plan, the requirement of locating closed-loop poles at invariant zeros would prevent from stabilizing the system. Positive invariance of $S(G,\rho)$ and stability of the closed-loop system cannot both be obtained.
Note also that if $GB$ is non-singular, $B \cap \text{Ker } G = \{0\}$.
(b) case $d > 0$.
If $\text{rank}(GB) = m - d$ ($= s - d$) with $d > 0$, $\text{rank}(NM) = n - m - d$, the zero pencil has "infinite elementary divisors". The sum of the degrees of the associated invariant polynomials is $d$.
Since the pencil is supposed to be regular, each zero-direction corresponds to only one zero. The set of all the generalized zero-directions associated to finite zeros spans an $(A,B)$-invariant subspace included in $\text{Ker } G$. This subspace is denoted $V$ and has dimension $n - m - d$.
The state zero directions associated to "infinite zeros" generate a subspace with dimension $d$ having no intersection with $V$.
For a regular pencil, condition $\text{Ker } NAM \cap \text{Ker } NM = 0$ is always satisfied.
"Infinite" zero-directions are such that $v = Mz$ and \[
\begin{cases}
NMz &= 0 \\
NAMz &\neq 0
\end{cases}
\]
Then, the subspace spanned by infinite zero-directions is precisely $\text{Ker } G \cap B$.
This subspace cannot be spanned by the solutions of (23) for finite values of $\lambda_i$.
Clearly, $NAM$ only admits $n - m - d$ finite invariant zeros and $n - m - d$ associated zero-directions defined by $n - m - d$ independent vectors $(z_1, \ldots, z_{n-m-d})$ solutions of (29).
\[
\square
\]
Cases of singular zero pencils: If $s < m$ or if $s = m$ but $\det(P(\lambda)) = 0 \ \forall \lambda \in \mathbb{C}$, then $P(\lambda)$ is said to be singular. In the last case, in order to have non-null solutions $z_i$ to the zero pencil equation (29) for any value of $\lambda_i$, we must necessarily have $\text{dim}(\text{Ker } NM) = d > 0$.
Note that in both cases, we have $s < m + d$.
Invariant zeros and associated invariant zero-directions are still associated to the elementary divisors of $P(\lambda)$, if such divisors still exist. To each invariant zero direction is associated a unique value of $\lambda \in \mathbb{C}$. The subspace of $\text{Ker } G$ spanned by the invariant zero-directions is the maximal $(A,B)$-invariant subspace of $\text{Ker } G$ not intersecting $B$. As it will be shown in the proof of proposition 3.3, the choice of invariant zeros as closed-loop eigenvalues remains necessary for obtaining the invariance of $\text{Ker } G$. 

From the same analysis as for proposition 3.2, it is then clear that the following restrictions also apply to the singular case:

- If the system has invariant zeros and if any of these invariant zeros is infinite, invariance of \( \text{Ker } G \) will not be obtained.
- If all the invariant zeros are finite but if any of them is unstable, invariance of \( \text{Ker } G \) and global stability cannot both be obtained with any constant state feedback.

But some other solutions of equation (23) may exist. They are associated to the column-minimal indices of \( P(\lambda) \) and are denoted ”controllable zero-directions”. These controllable zero-directions associated to a set of stable poles must be added to the set of invariant zero-directions, and assigned as closed-loop eigenvectors to obtain the invariance of \( \text{Ker } G \).

The decomposition of the zero pencil given in (30) indicates that the zeros of system \( S(A,B,G) \) are also the zeros of the non-proper system \( S((NAM)_1, (NAM)_2, (NAM)_3, (NAM)_4) \).

Any zero-direction associated to a value of \( \lambda \in \mathbb{C} \) can be defined from a vector \( z \in \mathbb{C}^{n-s} \) belonging to the transmission subspace \( \overline{\text{Tr}}(\lambda) \) of system \( ((NAM)_1, (NAM)_2) \). Here, by extension of the classical definition of (state) transmission subspaces previously stated (19), we include the state components and the input components in the definition of transmission subspace. \( \overline{\text{Tr}}(\lambda) \) is defined as the kernel of the pole pencil:

\[
[\lambda I_n - m - d - (NAM)_1 | - (NAM)_2]
\]

In order to satisfy relation (29) for some value of \( \lambda_i \), vector \( z_i \) should also belong to the kernel of matrix \( [-(NAM)_3 | - (NAM)_4] \)

It has to satisfy the two following relations:

\[
[\lambda_i I_n - m - d - (NAM)_1 | - (NAM)_2]z_i = 0
\]

\[
[-(NAM)_3 | -(NAM)_4]z_i = 0
\]

We are now ready to state the following proposition:

**Proposition 3.3**

A necessary and sufficient condition for \( \text{Ker } G \) to be spanned by zero-directions is:

\[
\delta = \dim \{ \text{Range} [-(NAM)_3 | -(NAM)_4]^T \} = 0
\]

This condition can be split into two alternatives:

- either \( d = 0 \)
- or \( d > 0 \) but \( [-(NAM)_3 | -(NAM)_4] = 0_{d,n-s} \) (as it is always the case, for instance, if \( A = -\alpha I_n \)).

Stability of the closed-loop system can also obtained only if all the invariant zeros of \( S(A,B,G) \) are stable (located in the left complex half-plane).
Invariant and controllable zeros have to satisfy the two relations (31) and (32). But since the second condition does not depend on the value of \( \lambda \), it constrains all the candidate vectors \( z \) to belong to \( \ker \begin{pmatrix} -NAM_1^1 & -NAM_2 \end{pmatrix} \).

If this kernel is not the whole space \( C^{n-s} \) but has dimension \( n - s - \delta \) with \( \delta > 0 \), then the associated zero-directions \( v = Mz \) can at most generate a subspace of \( \ker G \) with dimension \( n - s - \delta \).

**sufficiency**

Let us now assume that the condition above \( (\delta = 0) \) is satisfied. We can then suppress condition (32), and concentrate on relation (31). As noted above, the considered singular pencils satisfy \( s < m + d \). Then for any complex value of \( \lambda \), the transmission subspace \( \tilde{T}\lambda(\lambda) \) for system with state-matrix \( (NAM_1) \) and input matrix \( (NAM_2) \) has a dimension greater or equal to \( m + d - s \).

\[
\tilde{T}\lambda(\lambda) = \{ z; z \in C^{n \rightarrow r}; [\lambda I_{n-m-d} - (NAM_1) | -(NAM_2)]z = 0 \}
\]

(a) \( \dim[\tilde{T}\lambda(\lambda)] > m + d - s \) if \( \lambda_i \) is an invariant zero of \( S(A,B,G) \).

Note that the invariant zeros of \( S(A,B,G) \), when they exist, are the “input decoupling” zeros of system \( (NAM_1, NAM_2) \) (Kouvaritakis and Mac Farlane, 1976).

If the system \( (NAM_1, NAM_2) \) has \( p \) input decoupling zeros, these zeros are uncontrollable poles of \( (NAM_1) \) and the maximal controllability subspace of the pair \( (NAM_1, NAM_2) \) has dimension \( n - m - d - p \).

(b) \( \dim[\tilde{T}\lambda(\lambda)] = m + d - s \) if \( \lambda_i \) is simply any complex value which is not an invariant zero.

A sufficient condition for the existence of \( n - s \) zero-directions spanning \( \ker G \) is that the union of all the transmission spaces \( \tilde{T}\lambda(\lambda) \) (for at most \( n - s \) values of \( \lambda \)) span \( C^{n-s} \).

Any \( z \in C^{n-s} \) can be written \( z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \), with \( z_1 \in C^{n-m-d} \) and \( z_2 \in C^{m+d-s} \).

If it belongs to \( \tilde{T}\lambda(\lambda) \), it satisfies:

\[
(NAM_1)z_1 = \lambda z_1 - (NAM_2)z_2
\]

Under the assumption \( \delta = 0 \), the assignment of \( n - s \) closed-loop eigenvectors in \( \ker G \) can be obtained in two stages:

**stage 1**

First, apply a state feedback \( \Phi_1 \) to \( (NAM_1, NAM_2) \) to locate the \( n - m - d - s \) eigenvalues of matrix

\[
(NAM)_1' = (NAM)_1 + (NAM)_2\Phi_1
\]

in the stable region. These eigenvalues can be selected all different, and different from the uncontrollable poles (if there are any). In order for the set of vectors \( z_1 \) solutions of (33) to span \( \mathbb{R}^{n-m-d} \), the set of closed-loop poles, \( \lambda_i \), must include the uncontrollable poles of \( (NAM)_1' \) (and \( (NAM)_1 \)), with their order of multiplicity. As mentionned above, these uncontrollable poles are precisely the invariant zeros of \( S(A,B,G) \).

The eigenvectors of \( (NAM)_1' \) define \( n - m - d \) independent zero directions such that

\[
\begin{cases}
z_1 \neq 0 \\
z_2 = \Phi_1 z_1
\end{cases}
\]

The corresponding zero-directions obtained from these vectors through relation (27) span a subspace \( S_1 \) in \( \ker G \) with dimension \( n - m - d \).
Any solution of equation (33) is also a solution of
\[(NAM)' z_1 = \lambda z_1 - (NAM) z_2 \]
with \(z_2' = z_2 - \Phi_1 z_1\) \hspace{1cm} (34)

Controllability properties are invariant by state-feedback.
They are the same for \(((NAM)'_1, (NAM)_2)\) than for \(((NAM)_1, (NAM)_2)\).
And under the change of coordinates in (34), the \(n-s-d\) independent eigenvectors of \((NAM')_1\)
satisfy
\[
\begin{cases}
  z_1 \neq 0 \\
  z_2' = 0
\end{cases}
\] \hspace{1cm} (35)

Consider the Jordan form associated to the controllable subspace of \((NAM)'_1\), \(\Lambda = \Pi(NAM)'_1 \Gamma\),
where the ith line of \(\Pi\) is the left-eigenvector of \((NAM)'_1\) and the ith column of \(\Gamma\) the corresponding
right eigenvector for eigenvalue \(\lambda_i\), with all \(\lambda_i\) distinct by construction for \(i = 1, \ldots, n-m-d-p\).
Now, we can use a basis of \(\Re^{m+d-s}\) as \(m+d-s\) independent input vectors, \(z_2'i\), of \(m+d-s\) different
transmission subspaces \(Tr(\mu_i)\) for any set of selected distinct eigenvalues \((\mu_1, \ldots, \mu_{m+d-s})\).
For \(i = 1, \ldots, m+d-s\), each couple \((z_2'i, \mu_i)\) of an input vector and of a selected eigenvalue generates
a state vector of \(\Re^{n-m-d}\),
\[
z_{1i} = \Pi(\mu_i I - \Lambda)^{-1} \Gamma(NAM)_2 z_2'i
\]
By construction, all the vectors \(\begin{bmatrix} z_{1i} \\ z_2'i \end{bmatrix}\) are independent and independent from vectors satisfying
relation (35).

The associated coordinates of vectors \(z_i\) satisfying equation (33) are \(\begin{bmatrix} z_{1i} \\ z_2'i + \Phi_1 z_{1i} \end{bmatrix}\).
The corresponding zero-directions obtained from these vectors through relation (27) span a subspace
\(S_2\) of \(\text{Ker} \ G\) independent of \(S_1\), with dimension \(m+d-s\). Therefore, it is such that its direct sum
with \(S_1\) generates \(\text{Ker} \ G\):
\[
\text{Ker} \ G = S_1 \oplus S_2.
\]

\(\square\)

Note that the decomposition of \(\text{Ker} \ G\) used in this proof is far from being unique. In fact, as it
will be illustrated in the example, we have a free choice of the \(n-s-p\) controllable poles. And the
eigenstructure assignment problem can practically be solved in a single stage.
The formulation of proposition 3.3 includes the case of regular pencils, which was specifically des-
cribed by proposition 3.2.

\(\text{B - Spectral properties of } (A+BF)((\Re^n/\text{Ker} \ G))\)

Whenever the \((A,B)\)-invariance of \(\text{Ker} \ G\) is satisfied, the existence of positively invariant domains
\(S(G,\rho)\) in \(\Re^n\) for system (3) reduces to the existence of positively invariant domains \(S(I_{n-s},\rho)\)
in \(\Re^{n-s}\) for the restriction of \((A+BF)\) to \((\Re^n/\text{Ker} \ G)\), matrix \(F\) being constrained to be a
"friend" of \(\text{Ker} \ G\). Matrix \(H\) in equation (11) can precisely be interpreted as the map induced in
\((\Re^n/\text{Ker} \ G)\) by the map \((A+BF)\) in \(\Re^n\).

Let us now assume the \((A,B)\)-invariance of the subspace \(\text{Ker} \ G\), and wonder about the existence
positively invariant polyhedra \(S(D,\omega)\), with \(\text{Ker} \ D = \text{Ker} \ G\) and \(D\) to be constructed and not
necessarily full-rank. The following result can easily be shown.

**Proposition 3.4**

*Under the assumption that \( \text{Ker } G \) is \((A, B)\)-invariant and \( F \) a "friend" of \( \text{Ker } G \), the asymptotic stability of the restriction of \( A + BF \) to the quotient space \( \mathbb{R}^n / \text{Ker } G \) is a necessary and sufficient condition for the existence of a positively invariant polyhedral domain \( S(D, \omega) \) of system (3) such that \( \text{Ker } D = \text{Ker } G \).*

**Proof:**

From lemma 3.1, a necessary and sufficient condition for the \((A, B)\)-invariance of \( \text{Ker } G \) is the existence of a matrix \( H \in \mathbb{R}^{s \times s} \) such that:

\[
G(A + BF) = HG \tag{36}
\]

The dynamics of the projection of system (3) on \( \mathbb{R}^n / \text{Ker } G \) are described by the evolution of vector \( \beta \in \mathbb{R}^s \) such that

\[
\dot{\beta} = H\beta \tag{37}
\]

Kalman and Bertram (1960) have shown that asymptotical stability of a linear system (37) is equivalent to the existence of symmetrical positively invariant domains, \( v(\beta) \leq \delta \) with \( \delta > 0 \), associated to its quadratic Lyapunov functions, \( v(\beta) = \beta^T P \beta \), with \( P \) positive definite.

The square root of the Lyapunov function is a contracting norm in \( \mathbb{R}^s \) for the state of system (37).

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BF)|($\mathbb{R}^n / \text{Ker } G$) (Castelan and Hennet, 1991).

If the eigenvalues of $(A + BF)|($$\mathbb{R}^n / \text{Ker } G$), denoted $\mu_i + j\sigma_i$, are located in a specific stable region of the complex plane, then $J_2$ is a candidate matrix $H$ of lemma 3.1. This region of the complex plane is defined by the following relation:

$$\mu_i < -|\sigma_i|$$

The next proposition is based on this result.

**Proposition 3.5**

If the pair $(A, B)$ is controllable and $\text{Ker } G$ an $(A, B)$-invariant subspace, and under the conditions $\text{rank}(GB) = \text{rank}(G) = s \leq m$, there exists a positive vector $\rho$ such that $S(G, \rho)$ is positively invariant.

The existence of such a vector, $\rho$, is induced by the existence of a control matrix $F \in \mathbb{R}^{m \times n}$ such that:

1. $\text{Ker } G$ is an invariant subspace of system (3).
2. The real generalized eigenvectors of the restriction $(A + BF)|($$\mathbb{R}^n / \text{Ker } G$) span a subspace $R \subset \mathbb{R}^n$ such that $R \oplus \text{Ker } G = \mathbb{R}^n$. They can be selected as the column-vectors of a matrix $V_2$ satisfying:

$$GV_2 = I_s$$

(44)

The corresponding eigenvalues, $\mu_i + j\sigma_i$, satisfy $\mu_i < -|\sigma_i|$.

**Proof**

We can solve the eigenstructure assignment problem by the same decomposition technique as in Wonham (1985) (proposition 4.1, 88-89). Let $F_0$ be a friend of $\text{Ker } G$. The real generalized eigenvectors of $(A + BF_0)|($$\mathbb{R}^n / \text{Ker } G$) are the column-vectors of a matrix $V_1$ satisfying:

$$GV_1 = 0_{s \times (n-s)}$$

(43)

The real generalized eigenvectors associated to the eigenvalues of $(A + BF)|($$\mathbb{R}^n / \text{Ker } G$) span a subspace $R \subset \mathbb{R}^n$ such that $R \oplus \text{Ker } G = \mathbb{R}^n$. They can be selected as the column-vectors of a matrix $V_2$ satisfying:

$$GV_2 = I_s$$

(44)

The corresponding eigenvalues, $\mu_i + j\sigma_i$, satisfy $\mu_i < -|\sigma_i|$.

Consider the real Jordan canonical form of $(A + BF)$:

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

(45)

$J_1$ is the real Jordan canonical form of $(A + BF)|($$\mathbb{R}^n / \text{Ker } G$) and we have:

$$(A + BF)V_1 = V_1J_1$$

(46)

$J_2$ is the real Jordan canonical form of $(A + BF)|($$\mathbb{R}^n / \text{Ker } G$). The restriction of $(A + BF_0)$ to $(\mathbb{R}^n / \text{Ker } G)$ is denoted $\tilde{A}_0$ and defined by the canonical projection equation:

$$\tilde{A}_0G = G(A + BF_0)$$

(47)

Define $\tilde{B} = GB$. Under the assumption that $(A, B)$ is controllable, then $(\tilde{A}_0, \tilde{B})$ is also controllable in $(\mathbb{R}^n / \text{Ker } G)$.

The eigenvalues of $(\tilde{A}_0 + \tilde{B}F_1)$ can be selected so as to satisfy relations (42). Their associated
generalized real Jordan form is matrix $J_2$. Moreover, if $\text{rank}(GB) = \text{rank}(G) = s \leq m$, $s \leq m$, we can select as generalized real eigenvectors in $(\mathbb{R}^n/\ker G)$ the canonical basis constituting the columns of the identity matrix $I_s$. The corresponding generalized real eigenvectors in $\mathbb{R}^n$ are the column-vectors of matrix $V_2$ defined by relation (44).

To do so, it suffices to select $F_1$ such that:

$$BF_1 = J_2 - \bar{A}_0$$

(48)

And in particular,

$$F_1 = \bar{B}^T(B\bar{B}^T)^{-1}(J_2 - \bar{A}_0)$$

(49)

Let us now define matrix $G' \in \mathbb{R}^{(n-s)\times n}$ such that:

$$G' = [I_{n-s} \mid 0_{(n-s)\times s}][V_1 \mid V_2]^{-1}$$

(50)

And consider the nonsingular matrix $\begin{bmatrix} G' \\ G \end{bmatrix} \in \mathbb{R}^{n\times n}$. It is the inverse of matrix $[V_1 | V_2]$.

Then, we have:

$$\begin{bmatrix} G' \\ G \end{bmatrix}(A + BF) = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{bmatrix} G' \\ G \end{bmatrix}$$

(51)

And in particular,

$$G(A + BF) = J_2G$$

(52)

Relation (52) is equivalent to relation (11) when selecting $J_2$ as a candidate matrix $H$. Under conditions (42) on the eigenvalues of $J_2 = -s_0I_s + K$, $s_0 > 0$ can be selected so that $(s_0I_s - |K|)$ is an M-matrix (Castelan and Hennet, 1991). Then, from a classical result (Poole and T. Bouillon, 1974), this property implies the existence of a positive vector $\omega$ such that $(-s_0I_s + |K|)\omega \leq 0$. Then, from proposition (2.2), the polyhedron $S(G, \omega)$ is a positively invariant set of the closed-loop system.

$\square$

If matrix $GB$ is not full-rank, a different construction should be used. The only case of possible invariance of $\ker G$ is presented in proposition 3.3. It is the "pathological case":

$$[(NAM)_3](NAM)_4 = 0_{d,n-s}.$$  

In this case, system $(\bar{A}_0, \bar{B})$ is still controllable, but the column-vectors $e_i$ of matrix $E$ in relation $GV_2 = E$ cannot be freely chosen. They have to satisfy:

$$e_i = (\lambda_iI_s - GAG^\times)^{-1}GBw_i$$

(53)

with $w_i$ arbitrarily selected in $\mathbb{R}^n$ with $w_i \notin \ker(GB)$, $\lambda_i$ not an eigenvalue of $A$, and $G^\times$ such that $GG^\times = I_s$.

Then, under condition (32), $s$ non-singular real vectors $e_i$ can be obtained for $s$ different eigenvalues $\lambda_i$. They define a non-singular matrix $E$ such that $GV_2 = E$, and $G(A + BF) = EJ_2E^{-1}G$.

Under spectral conditions (42), there exist a positive vector $\rho$ such that $S(E^{-1}G, \rho)$ is invariant, and $\ker(E^{-1}G) = \ker G$, but in general it does not imply the invariance of a domain $S(G, \rho)$. That is the reason of the restrictive assumption $\text{rank}(GB) = s$ in proposition 3.5.
The purpose of the original problem is to obtain the positive invariance of the domain $S(G, \rho)$, for which vector $\rho$ is given. We can directly use the spectral assignment method described in proposition 3.5. But in this case, each stable complex closed-loop eigenvalue assigned in $(\mathbb{R}^n/\ker G)$ is constrained to belong to a region of the complex plane included in the region defined by relation (42).

Let $J_2$ be the candidate matrix $H$ and consider the spectral assignment described in the proof of proposition 3.5. Under relation (52), the rows of matrix $G$, denoted $g_i$, are the left generalized real eigenvectors associated to $J_2$. Let $\rho_i$ be the component of vector $\rho$ associated to $g_i$, such that:

$$\rho_i \leq g_i x \leq \rho_i$$

for $i = 1, \ldots, s$ (54)

For a particular pair of simple complex eigenvalues $\mu_i \pm j \sigma_i$, associated to left real eigenvectors $g_i$ and $g_{i+1}$, we can write:

$$\begin{bmatrix} \mu_i & -\sigma_i \\ \sigma_i & \mu_i \end{bmatrix} \begin{bmatrix} g_i \\ g_{i+1} \end{bmatrix} = \begin{bmatrix} g_i \\ g_{i+1} \end{bmatrix} (A + BF)$$

Under condition (42), $(s_0 I_s - |K|)$ is a $M$-matrix. Then, $s_0 \geq r(|K|)$, where $r(|K|)$ is the spectral radius of $|K|$ (Poole and Bouillon, 1974). This implies that:

$$s_0 \geq -\mu_i$$

(56)

Therefore, from the block diagonal form of $J_2$, we must have:

$$\begin{bmatrix} -\mu_i & -|\sigma_i| \\ -|\sigma_i| & -\mu_i \end{bmatrix} \begin{bmatrix} \rho_i \\ \rho_{i+1} \end{bmatrix} \geq 0$$

And the above inequality should be strict to get asymptotic stability. Then, $\mu_i \pm j \sigma$ satisfy:

$$\mu_i < -\left( \frac{\rho_{i+1}}{\rho_i} \right) |\sigma_i|, \text{ if } \rho_i \geq \rho_{i+1}$$

(57)

$$\mu_i < -\left( \frac{\rho_i}{\rho_{i+1}} \right) |\sigma_i|, \text{ otherwise}$$

(58)

In the case of simple real eigenvalues, this kind of restriction does not appear. If $\lambda_i$ and $\lambda_{i+1}$ are real negative and distinct, the following inequality is satisfied for any positive value of $\rho_i$, $\rho_{i+1}$:

$$\begin{bmatrix} -\lambda_i & 0 \\ 0 & -\lambda_{i+1} \end{bmatrix} \begin{bmatrix} \rho_i \\ \rho_{i+1} \end{bmatrix} \geq 0$$

Therefore, under the assumption of controllability of the pair $(A,B)$, it is always possible to select all the closed-loop eigenvalues in $(\mathbb{R}^n/\ker G)$ real negative and distinct, so that the assignment scheme described above can be used independently from vector $\rho$. The components of this vector simply have to be strictly positive.

**AN EIGENSTRUCTURE ASSIGNMENT ALGORITHM**

As shown in the preceding section, positive invariance of $S(G, \rho)$ for the closed-loop system (3)
can be derived from the two following conditions:

1. Assignment of \((n - s)\) generalized real eigenvectors in \(\text{Ker } G\).
2. Assignment of \(s\) generalized real eigenvectors in a subspace \(R\) such that \(R \oplus \text{Ker } G = \mathbb{R}^n\), under appropriate conditions (57) or (58) on the associated eigenvalues.

If we assume \((A,B)\) controllable and \(\text{rank}(GB) = \text{rank}(G) = s \leq m\), from proposition 3.4 and the discussion above, the second requirement can always be met once the first one has been satisfied. Then, from propositions 3.2 and 3.3, these assumptions are sufficient for the existence and computation of a state feedback for which the closed-loop system lets \(S(G, \rho)\) positively invariant.

The \((n - s)\) (possibly not distinct) eigenvalues related to \(\text{Ker } G\), \(\lambda_i\), and their associated directions, \(v_i\) and \(w_i\), satisfy:

\[
P(\lambda_i) \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  

(59)

Relatively to \((\mathbb{R}^n / \text{Ker } G)\), the selected eigenvalues, \(\lambda_i'\), and associated directions, \(v_i'\) and \(w_i'\), satisfy:

\[
P(\lambda_i') \begin{bmatrix} v_i' \\ w_i' \end{bmatrix} = \begin{bmatrix} 0 \\ e_i \end{bmatrix}, \text{ with } e_i = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i.
\]  

(60)

To compute the eigenvectors of the closed-loop system, we use Moore’s formulation of the pole pencil \(S(\lambda_i)\):

\[
S(\lambda_i) = [\lambda_i I - A | -B]
\]  

and of its kernel: \(K_{\lambda_i} = \begin{bmatrix} N_{\lambda_i} \\ M_{\lambda_i} \end{bmatrix}\)  

(61)

Then, any solution \(v_i \in \mathbb{R}^n\) can be written, for some vector \(k_i\) of appropriated dimension,

\[
v_i = N_{\lambda_i}k_i, \text{ with } Fv_i = M_{\lambda_i}k_i
\]  

(62)

If the eigenvector belongs to \(\text{Ker } G\), it should satisfy:

\[
GN_{\lambda_i}k_i = 0
\]  

(63)

The \(s\) other eigenvectors are selected so as to satisfy:

\[
GN_{\lambda_i}k_i = e_i
\]  

(64)

These formulations can be directly used for simple real eigenvalues. In the case of simple complex conjugate eigenvalues, \((\mu_i + j\sigma_i, \mu_i - j\sigma_i)\), the complex kernels \((K_{\mu_i+j\sigma_i}, K_{\mu_i-j\sigma_i})\) are replaced by the associated real kernels of the real pencils defined as follows.

The closed-loop eigenvectors and controls associated to \(\mu_i + j\sigma_i\) and \(\mu_i - j\sigma_i\) (with \(\sigma_i \neq 0\)) are denoted: \(\begin{bmatrix} v_i \\ w_i \end{bmatrix}, \begin{bmatrix} v_i^* \\ w_i^* \end{bmatrix}\). They are complex conjugate.
The corresponding real directions, denoted \[\begin{bmatrix} v_{\mu i} \\ w_{\mu i} \end{bmatrix}, \begin{bmatrix} v_{\sigma i} \\ w_{\sigma i} \end{bmatrix}\] are defined as the real and imaginary parts of \[\begin{bmatrix} v_i \\ w_i \end{bmatrix}:\]
\[
\begin{bmatrix} v_{\mu i} \\ w_{\mu i} \end{bmatrix} = \frac{1}{2} \left\{ \begin{bmatrix} v_i \\ w_i \end{bmatrix} + \begin{bmatrix} v_i^* \\ w_i^* \end{bmatrix} \right\}, \quad \begin{bmatrix} v_{\sigma i} \\ w_{\sigma i} \end{bmatrix} = \frac{-j}{2} \left\{ \begin{bmatrix} v_i \\ w_i \end{bmatrix} - \begin{bmatrix} v_i^* \\ w_i^* \end{bmatrix} \right\}
\]

For a pair of simple complex conjugate eigenvalues of \((A + BF)|Ker G\), the real (state and input) directions satisfy the following equation:

\[
\begin{bmatrix} \mu_i I_n - A & -\sigma_i I_n & -B & 0_{n\times m} \\ \sigma_i I_n & \mu_i I_n - A & 0_{n\times m} & -B \\ G & 0_{n\times m} & 0_{n\times m} & 0_{n\times m} \\ 0_{n\times n} & G & 0_{n\times m} & 0_{n\times m} \end{bmatrix} \begin{bmatrix} v_{\mu i} \\ v_{\sigma i} \\ w_{\mu i} \\ w_{\sigma i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

And for a pair of simple complex conjugate eigenvalues of the restriction of \((A+B\mathcal{F})\) to \((\mathbb{R}^n/Ker G)\), they satisfy:

\[
\begin{bmatrix} \mu_i I_n - A & -\sigma_i I_n & -B & 0_{n\times m} \\ \sigma_i I_n & \mu_i I_n - A & 0_{n\times m} & -B \\ G & 0_{n\times m} & 0_{n\times m} & 0_{n\times m} \\ 0_{n\times n} & G & 0_{n\times m} & 0_{n\times m} \end{bmatrix} \begin{bmatrix} v'_{\mu i} \\ v'_{\sigma i} \\ w'_{\mu i} \\ w'_{\sigma i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e_i \\ e_{i+1} \end{bmatrix}
\]

Let \(V = [V_1 \mid V_2]\) be the matrix of the desired real generalized eigenvectors, and \(W = [W_1 \mid W_2]\) the associated input directions. The feedback gain matrix providing the desired eigenstructure assignment is:

\[F = WV^{-1}\]

In a classical way (Kautsky, Nichols and Van Dooren, 1985), the feedback operator may also be directly computed by:

\[F = R_0^{-1}Q_0^T(VJV^{-1} - A)\]

where \(B = [Q_0 \quad Q_1] \begin{bmatrix} R_0 \\ 0 \end{bmatrix}\), with \(Q = [Q_0 \quad Q_1]\) orthogonal and \(R_0\) non-singular.

**EXEMPLE**

Consider the following data:

\[
A = \begin{bmatrix} -3.2605 & 0.8207 & -12.4138 & 14.9777 \\ -4.2012 & 8.6924 & -29.9610 & 12.1320 \\ 3.2264 & -11.9363 & 5.8227 & 19.5983 \\ -4.7162 & 9.8930 & -19.7593 & 0.0005 \end{bmatrix}, \quad B = \begin{bmatrix} 1.7286 & 0.0000 & 3.4339 \\ 0.0000 & 1.9171 & 2.9449 \\ 2.6485 & -2.5000 & 4.6522 \\ -3.3557 & 2.0874 & 0.0000 \end{bmatrix}
\]

The open-loop system has eigenvalues:

\[
\lambda(A) = \begin{bmatrix} -2.5660 \\ 9.7752 \\ 2.0230 \pm j4.3426 \end{bmatrix}
\]
The state constraints are defined by

\[ G = \begin{bmatrix} 2.0000 & -4.7800 & 9.5000 & 0.0000 \\ 0.5000 & 0.0000 & 3.8300 & -4.2200 \end{bmatrix} \quad \text{and} \quad \rho = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \]

System \( S(A, B, G) \) has a stable zero \( \lambda_1 = -2.7936 \). This value and \( \lambda_2 = -1.0000 \) are selected as closed-loop eigenvalues, for which the associated eigenvectors span \( \ker G \). The 2 other poles have been chosen as follows:

\[ \lambda_1' = -2.0000 + j1.0000, \quad \lambda_2' = -2.0000 - j1.0000 \]

Then, we get

\[
A_0 = \begin{bmatrix} -0.8046 & -0.0963 & -0.2133 & 0.0946 \\ 2.7947 & -2.2078 & -0.1826 & -1.5927 \\ 1.2072 & -0.0843 & -1.6438 & -1.2655 \\ 1.7112 & -1.2206 & 2.5492 & -3.1374 \end{bmatrix}
\]

\[
V = \begin{bmatrix} -0.0137 & 0.0557 & -0.0585 & -0.5015 \\ 0.1576 & 0.3738 & -0.7392 & -0.7296 \\ 0.1875 & 0.1764 & -0.3596 & -0.2615 \\ 0.1685 & -0.0703 & -0.3333 & -0.2968 \end{bmatrix}
\]

and

\[
F = \begin{bmatrix} -0.6334 & 0.0581 & -0.7332 & 1.2010 \\ 2.0608 & -5.2307 & 9.5084 & 0.4274 \\ 1.0340 & -0.2963 & 3.9221 & -4.9388 \end{bmatrix}
\]

On Fig. 1, we can see the stable time evolution of \( x_1' \) and \( x_2' \), which are the coordinates of the state vector in an orthonormal basis of the plane spanned by the 2 column-vectors of \( V_2 \), for two different initial conditions:

\[
x_{01} = [0.0443 \quad 0.0460 \quad 0.2833 - 0.2115]^T \quad \text{(trajectory ——)} \quad \text{and} \quad x_{02} = [0.0709 \quad 0.3396 \quad 0.1300 \quad 0.6003]^T \quad \text{(trajectory - - )}.
\]

For any admissible initial state, the state trajectories converge to the zero-state without violating the constraints.

**CONCLUSION**

Many systems are subject to constraints which can be represented as linear inequality constraints on the state vector all along its trajectory. If these constraints are time invariant, a natural way of solving the corresponding constrained regulation problem is to construct a stabilizing state feedback for which the closed-loop system lets invariant the polyhedron \( S(G, \rho) \) bounded by the constraints. Some algebraic invariance conditions have been obtained for continuous-time linear systems. These conditions have then been interpreted in terms of structural properties for the triplet \( (A,B,G) \). In particular, the kernel of the map \( G \) has to be made invariant by the control, and this property requires to locate a sufficient number of closed-loop poles at the zeros of the system. This assignment is impossible if the system has infinite invariant zeros, and it is not consistent with the stability requirement if any invariant zero is unstable. Therefore, computation of the system zeros readily provides an answer to the existence of a stabilizing invariant regulator. The other poles of the
Figure 1: Trajectories in projection

closed-loop system should be located in a particular region of the left-half complex plane. If all these poles are chosen real negative and distinct, the placement algorithm does not depend on the magnitude of the constraints (which are supposed symmetrical), as long as the zero-state is strictly included in the feasible domain (assumption \( \rho > 0 \)). Once the appropriate closed-loop eigenvalues and eigenvectors have been selected, a satisfactory gain matrix can be directly computed.

REFERENCES


