Stability and Stabilization of Delay Differential Systems*

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Conditions for respect of linear constraints and for asymptotic stability of linear delay differential systems with bounded additive disturbances are obtained from a characterization of positive invariance properties for these systems.

Key Words—Delay differential equations; positive invariance; asymptotic stability.

Abstract—The delay systems considered here are represented by linear delay differential equations. The system parameters and the delays are assumed to be imperfectly known. The instantaneous state vector is perturbed by a bounded external disturbance vector. The problem addressed is that of characterizing conditions that guarantee that the trajectory of the instantaneous state vector remains in a domain defined by a set of symmetrical linear constraints. It is shown that the positive invariance property can be used to solve this problem, and that positive invariance of a compact domain of the instantaneous state space implies delay-independent asymptotic stability of the associated deterministic system. The possible use of these results for the control of a multiple-delay MIMO differential model is then presented. Finally, an example is given.

1. INTRODUCTION

Delays are inherent in many physical and technological systems. In particular, pure delays are often used to ideally represent the effects of transmission, transportation and inertia phenomena. Delay differential equations constitute basic mathematical models of real phenomena, for instance in biology, mechanics and economics. Stability of delay systems is an important issue addressed by many authors and for which surveys can be found in several, monographs (see e.g. Lakshmikantham and Leela, 1969; Hale, 1977; Stépan, 1989; Lafay and Conte 1995). In particular, some delay-independent stability conditions have been proposed for linear delay differential equations (Kamen, 1982, 1983). Delay-independent conditions are particularly interesting for uncertain systems when robust stability is requested. But they may be too restrictive, as noted by several authors (Cheres et al., 1989; Niculescu et al., 1994; Chen, 1995).

On the other hand, it is also important to use available knowledge on delays and delay terms for control purposes. The objective of this paper is to obtain stability conditions that provide better insights into the effects of delay terms on the system behavior, and to use these conditions and insights in control problems.

This paper deals with a perturbed autonomous delay differential system whose instantaneous state vector is constrained to belong to a polyhedral domain that may correspond to the admissible running conditions of the system. Some simple algebraic conditions are established to characterize the positive invariance of this domain. It is also shown that if the polyhedral domain is a compact set, its positive invariance implies asymptotic stability of the unperturbed system. The stability condition obtained are independent of the delay values, and are generally less restrictive than those previously presented in the literature (Klai et al., 1994; Niculescu et al., 1994).

Notation. The following notation is used throughout the paper. The time derivative and the transpose of a vector \( y(t) \) are respectively denoted by \( y'(t) \) and \( y^T(t) \). The matrix \( L \) is the identity matrix in \( \mathbb{R}^{n \times n} \); the vector \( 1_n \) is defined as \( 1_n = [1 \ldots 1]^T \in \mathbb{R}^n \) and \( 0_n = [0 \ldots 0]^T \in \mathbb{R}^n \).
The matrix $\text{Diag}(\beta_i)$, for $i = 1, \ldots, g$, is the diagonal matrix with components $\beta_i$. For any matrix $M \in \mathbb{R}^{n \times m}$, $M_i$ denotes the $i$th row vector of $M$, and $M_{ji}$ denotes the matrix $M$ in which the row vector $M_i$ has been deleted. $|M|$ denotes the matrix of the absolute values of the components of $M$: $|M_{yi}| = |M_{yi}|$. The matrix $\bar{M}$ is defined by $\bar{M}_{ii} = M_{ii}$ and $\bar{M}_{ji} = |M_{ji}|$ for $j \neq i$. Classically the infinity norm of $M$ is defined by $\|M\|_\infty = \max_{i=1,\ldots,n} \sum_{j=1}^m |M_{ji}|$. Inequalities between vectors and inequalities between matrices are componentwise. $\mathbb{R}_+$ denotes the set of non-negative vectors of $\mathbb{R}^n$. $C_\mathbb{T} = C([-\tau, 0], \mathbb{R}^m)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^m$ with the topology of uniform convergence. $\|\cdot\|_\infty$ stands for the norm of a function $\varphi \in C_\mathbb{T}$. We denote by $C_{\mathbb{T}}^+$ the set defined by $C_{\mathbb{T}}^+ = \{\varphi \in C_\mathbb{T} : \|\varphi\|_\infty < \nu\}$, where $\nu$ is a positive real number.

2. PROBLEM PRESENTATION

2.1. A differential model for linear delay systems

A classical way of representing an autonomous delay system is through use of a delay differential equation (Hale, 1977; Cheres et al., 1989; Niculescu et al., 1994):

$$\dot{y}(t) = A_0y(t) + \sum_{i=1}^p A_i y(t - \tau_i) + Ew(t) \quad \text{for} \ t \geq t_0$$

(1)

where $y(t) \in \mathbb{R}^n$ is the instantaneous state vector and $w(t) \in \mathbb{R}^r$ a bounded external disturbance vector. The system 'delays' $\tau_i$, for $i = 1, \ldots, p$, are positive. Their maximum value is $\tau = \max_{i=1,\ldots,p} \tau_i$.

The model (1) is well-defined under the initial conditions

- an initial time value $t_0 \in \mathbb{R}_+$;
- a continuous vector function $\varphi(\cdot)$ defined on $[-\tau, 0]$ and belonging to $C_{\mathbb{T}}^+$; it characterizes the initial trajectory of system (1) through

$$y(t_0 + \theta) = \varphi(\theta) \quad \forall \theta \in [-\tau, 0],$$

(2)

By differentiability, the solution of (1) is continuous for $t \geq t_0$. The system (1) is supposed to have uncertain delays $\tau_i$, $i = 1, \ldots, p$, and uncertain entries in $A_i$, $i = 0, \ldots, p$.

2.2. Trajectory constraints

The system (1) is required to satisfy linear constraints on its instantaneous state vector. These constraints are symmetrical, defined by a matrix $G \in \mathbb{R}^{n \times m}$ and a positive vector $\mu \in \mathbb{R}^m$, and therefore describe a symmetrical polyhedron of admissible instantaneous states, denoted by $S(G, \mu)$:

$$S(G, \mu) = \{y \in \mathbb{R}^n : -\mu \leq Gy \leq \mu\}.$$ 

(3)

The constraints should be satisfied along the system trajectories for all the possible positive values of the delays $\tau_i$, $i = 1, \ldots, p$. Hence, assuming that instantaneous state constraints are satisfied by initial conditions $\varphi(.)$,

$$-\mu \leq G\varphi(\theta) \leq \mu \quad \forall \theta \in [-\tau, 0],$$

(4)

what are the conditions under which the condition (4) implies

$$-\mu \leq Gy(t) \leq \mu \quad \forall t \geq t_0?$$

(5)

A norm-boundedness condition applies to the external disturbance vector $w(t)$. In this study, since we are mainly interested in the infinity norm, the condition takes the form

$$\|Gw(t)\|_\infty \leq \omega \quad \forall t \geq t_0$$

(6)

2.3. The positive invariance approach

A natural way of maintaining the system trajectory in $S(G, \mu)$ is to impose the positive invariance of $S(G, \mu)$ with respect to the system (1). By definition, this corresponds to the following implication

$$\forall t \geq t_0, \forall \theta \in [-\tau, 0], \ -\mu \leq Gy(t + \theta) \leq \mu \Rightarrow -\mu \leq Gy(s) \leq \mu \quad \forall s \geq t.$$ 

(7)

2.3.1. The extended Farkas lemma. A basic tool for characterizing positive invariance conditions with respect to linear dynamical systems is the extended Farkas lemma (Hennet, 1989), in the version suitable for continuous-time systems (Castelan and Hennet, 1993). Consider matrices $P \in \mathbb{R}^{p \times n}$, $Q \in \mathbb{R}^{r \times n}$, $R \in \mathbb{R}^{m \times n}$ and vectors $\phi \in \mathbb{R}^n$, $\chi \in \mathbb{R}^r$, $\psi \in \mathbb{R}^m$. The extended Farkas lemma is stated as follows.

**Lemma 1.** (Hennet (1989).) The system of inequalities $Px \leq \psi$ is satisfied by any point of the non-empty convex polyhedral set defined by the system of constraints

$$Qx \leq \phi,$$

$$Rx = \chi$$

if and only if there exist a (dual) matrix $U \in \mathbb{R}^{p \times q}$, with non-negative coefficients and a
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(dual) matrix \( V \in \mathbb{R}^{p \times r} \), satisfying the following conditions;

\[
UQ + VR = P, \tag{8}
\]

\[
U\phi + V\chi \leq \psi. \tag{9}
\]

This lemma can be considered as an extension of the well-known Farkas lemma (Schrijver, 1987) to the matrix case.

2.3.2. Delay differential systems. To facilitate the use of Lemma 1, the domain \( S(G, \mu) \) can be rewritten equivalently as

\[
R[J, \gamma] = \{y \in \mathbb{R}^n : Jy(t) \geq \gamma\}, \tag{10}
\]

with

\[
J = \begin{bmatrix} G \\ -G \end{bmatrix}, \quad \gamma = \begin{bmatrix} \mu \\ -\mu \end{bmatrix}.
\]

Given the assumptions (6) on the domain of admissible perturbations, the positive invariance of the polyhedron \( S(G, \mu) \) with respect to the system (1) can be characterized as follows.

Proposition 1. A necessary and sufficient conditions for the positive invariance of \( S(G, \mu) \) with respect to the system (1) for any external disturbance \( w(t) \) such that \( \|GEw(t)\|_x \leq \omega \) is the existence of \( p + 2 \) real matrices \( (H_i \in \mathbb{R}^{r \times r}) \) for \( i = 0, \ldots, p + 1 \) such that

\[
H_0G = GA, \tag{11}
\]

\[
H_{p+1}GE = GE, \tag{12}
\]

\[
\left( \hat{A}_0 + \sum_{i=1}^p |H_i| \right) \mu + |H_{p+1}| \omega 1_g \leq 0_g, \tag{13}
\]

Proof. For the implication (7) to be valid at all the points of the admissible instantaneous state domain \( S(G, \mu) \), it is necessary and sufficient to guarantee the admissibility of any infinitesimal motion starting from any point of any facet of this domain. Under the assumptions on the function of initial conditions \( \phi \) and on the disturbance vector \( w(t) \), it can be derived from (Seifert, 1976, Corollary 1, p. 295) that a necessary and sufficient condition for positive invariance of \( S(G, \mu) \) with respect to the system (1) is given by the following implication

\[
\begin{align*}
(C1) & \quad Jy(t - \tau_k) \leq \gamma \quad \text{for } k = 1, \ldots, p, \\
(C2) & \quad Jy(t) \leq \gamma_j \quad \text{for } j \in (1, \ldots, 2g) - \{i\}, \\
(C3) & \quad J\tilde{y}(t) = \gamma_i \\
\Rightarrow & \quad J\tilde{y}(t) \leq 0 \quad \exists t = 1, \ldots, 2g, \forall t \geq t_0. \tag{14}
\end{align*}
\]

And, given (1), the resulting assertion of the implication (14) can be replaced equivalently by

\[
J[A_0y(t) + \sum_{i=1}^p A_iy(t - \tau_i) + Ew(t)] \leq 0. \tag{15}
\]

Then lemma 1 can be repeatedly applied for

\[
i = 1, \ldots, 2g
\]

by setting

\[
x = \begin{bmatrix} y(t) \\ y(t - \tau_1) \\ \vdots \\ y(t - \tau_p) \\ w(t) \end{bmatrix} \in \mathbb{R}^{(p+1)\times 1},
\]

\[
P = J[A_0 \quad A_1 \quad \ldots \quad A_p \quad E] \in \mathbb{R}^{1 \times (p+1)\times 2},
\]

\[
Q = \begin{bmatrix} J_{1:1} & 0 & \ldots & 0 \\ J_{2:1} & J & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix},
\]

\[
R = J[y(t)],
\]

\[
\phi = \begin{bmatrix} \gamma_{l:1} \\ \gamma \\ \gamma_{l:2g} \end{bmatrix} \in \mathbb{R}^{(2g(p+2) - 1)},
\]

\[
\psi = 0, \quad \chi = \gamma_i.
\]

Hence the set of implications (14) for \( i = 1, \ldots, 2g \) is equivalent to the existence of \( p + 1 \) non-negative matrices \( \mathbb{K}_k \in \mathbb{R}^{2g \times 2g} \) and one matrix \( \mathbb{K}_0 \in \mathbb{R}^{2g \times 2g} \) with non-negative off-diagonals, satisfying

\[
\mathbb{H}_0 J = JA_0, \tag{16}
\]

\[
\begin{bmatrix} \mathbb{K}_1 & \ldots & \mathbb{K}_{p+1} \end{bmatrix} \begin{bmatrix} \gamma \\ \gamma_j \\ \gamma_{l:2g} \end{bmatrix} \leq 0_{2g}. \tag{18}
\]

Each matrix \( \mathbb{K}_k \), for \( k = 0, \ldots, p + 1 \), can be decomposed into four \( g \times g \) blocks as follows:

\[
\mathbb{K}_k = \begin{bmatrix} H_k^1 & H_k^2 \\ H_k^{21} & H_k^2 \end{bmatrix}.
\]

The necessity of the conditions (11)–(13) for positive invariance of \( S(G, \mu) \) with respect to the system (1) is then established for matrices:

\[
H_k = H_k^1 - H_k^2 \quad \text{for } k = 0, \ldots, p + 1.
\]

The conditions (11) and (12) are directly derived from (17), and the condition (13) is obtained from (18) using the positivity of the vector \( \mu \) and of the scalar \( \omega \) and the term-to-term minorations

\[
\hat{A}_0 \leq H_k^1 + H_k^2, \quad |H_k| \leq H_k^1 + H_k^2 \quad \text{for } k = 1, \ldots, p + 1.
\]
Conversely, the sufficiency of the conditions (11)–(13) is shown by selecting
\[ \mathcal{K}_t = \begin{bmatrix} H_k^+ & H_k^- \\ H_k^- & H_k^+ \end{bmatrix}, \]
with, by definition,
\[ H_k^{+i,j} = \max (H_{kij}, 0), \quad H_k^{-i,j} = \max (-H_{kij}, 0) \]
for \( k = 1, \ldots, p + 1, \) and
\[ H_k^{+i,j} = \max (H_{0ij}, 0) \]
for \( j \neq i, \)
with, by definition,
\[ H_k^{+0,j} = 0. \]
Then it is easy to check that the \( p + 1 \) non-negative matrices \( \mathcal{K}_t \in \mathbb{R}^{2g \times 2g} \) and the matrix \( \mathcal{K}_0 \in \mathbb{R}^{2g \times 2g} \) with non-negative off-diagonal terms satisfying the conditions (16)–(18), which are equivalent to positive invariance of \( S(G, \mu) \) with respect to the system (1).

If there exists a set of matrices \( H_i \) satisfying the relations (11)–(13) then a feasible solution can be obtained by linear programming. The condition (13) can be put into a linear form (Hennet, 1989; Castelan and Hennet, 1993) and a linear objective function can be selected. For instance, at this stage, the right-hand side of (18), \( \omega 1_g \), can be replaced by \( p1_g \), with \( p \) the objective function to be minimized. Positive invariance is then obtained if the optimal value of \( p \) is non-positive.

Each matrix \( H_0, H_i, i = 1, \ldots, p, H_{p+1}, \) can be seen as a dual matrix respectively associated with constraints \( -\mu \leq G_y(t) \leq \mu, \) with \( \mu \) the objective function to be minimized. Positive invariance is then obtained if the optimal value of \( \mu \) is non-positive.

2.4. Positive-invariance relations with normalized bounds

By appropriate scaling of the rows of the matrix \( G \), it is always possible, for any strictly positive vector \( \omega \), to impose \( 1_g \) as the vector of bounds in the constraints (5). It suffices to define a matrix \( \tilde{G} \) by \( \tilde{G} = \text{diag} \left( \frac{1}{\mu_j} \right) G \). The polyhedral domain \( S(G, \mu) \) is identical to \( S(\tilde{G}, 1_g) \). Application of Proposition 1 to the domain \( S(\tilde{G}, 1_g) \) yields the following equivalent result.

Corollary 1. Positive invariance of \( S(\tilde{G}, 1_g) \) with respect to the system (1) for any external disturbance \( w(t) \) such that \( -d \leq \tilde{G} E w(t) \leq d \), with \( d = \text{Diag} \left( \frac{1}{\mu_i} \right) \omega 1_g \), is equivalent to the existence of \( p + 2 \) real matrices \( (H_{i} \in \mathbb{R}^{g \times g}) \), for \( i = 0, \ldots, p + 1, \) such that
\[ H_{i}^T G' = G' A_i, \]
\[ H_{i+1}^T G'E = G'E, \]
\[ (H_{0}^T + \sum_{i=1}^{g} |H_{i}|) \mu \leq \omega 1_g. \]

This corollary is derived from Proposition 1, applied to the unit polytope \( S(\mu, 1_g) \). To obtain positive invariance of \( S(\mu, 1_g) \), the only possible
choice for matrices $H_i$, for $i = 0, \ldots, m$, is $H_i = A_i$.

Note that the invariance condition (13) of Proposition 1 requires knowledge of the matrices $A_i$, because it is expressed in terms of the matrices $H_i$ computed from (11). In contrast, the condition (28) is expressed directly using the matrices $A_i$, and may thus be established without precisely knowing these matrices. Corollary 2 fits in well with the robustness issue, since it allows for uncertainties about delays and the system parameters.

3. STABILITY CONDITIONS

The deterministic system associated with the system (1) is

$$\dot{y}(t) = A_0 y(t) + \sum_{i=1}^{m} A_i y(t - \tau_i). \quad (29)$$

3.1. Stability conditions via positive invariance

Stability conditions of the system (29) are now investigated, under the assumptions $G \in \mathbb{R}^{n\times m}$, $g \geq n$, and $\text{rank}(G) = n$. By definition, the system (29) is said to be asymptotically stable if, for some vector norm, $\|y(t)\| \to 0$ as $t \to \infty$.

The robust stability linked to the positive invariance of compact polyhedra is presented in the following theorem.

**Theorem 1.** If there exists $G \in \mathbb{R}^{n\times m}$, $g \geq n$, and $\text{rank}(G) = n$, such that $\exists H_i \in \mathbb{R}^{k\times s}$, for $i = 0, \ldots, m$, satisfying

$$H_i G = G A_i, \quad (30)$$

then

(i) $S(G, 1_g)$ is positively invariant with respect to the system (29);

(ii) the asymptotic stability of the system (29) is obtained; it is independent of the values of the delays $\tau_i$ for $i = 1, \ldots, m$.

**Proof.** The relations (30) and (31) express the positive invariance of $S(G, 1_g)$ with respect to the system (29). This domain is denoted equivalently by $K[J, 1_{2g}]$, with $J = \left[ \begin{array}{c|c} G & -G \\ \hline \end{array} \right]$. Consider the function

$$v(y) = \|G y\| = \max_{j=1, \ldots, g} |(G y)_{j}| = \max_{j=1, \ldots, 2g} (J y)_{j}. \quad (32)$$

For rank $(G) = n$, this function is a polyhedral norm in $\mathbb{R}^n$, denoted by $\| \cdot \|_G$. Consider its value at time $t$:

$$v(y(t)) = \|y(t)\|_G = \xi, \quad \text{with } \xi > 0.$$ 

Following Razumikhin's theorem (Hale, 1977), it is assumed that the following inequality holds for any positive number $\delta > 1$

$$v(y(t + \theta)) < \delta v(y(t)) \quad \forall \theta \in [-\delta, 0). \quad (33)$$

The condition (33) implies

$$J_i y(t + \theta) < \delta \xi, \quad \forall j = 1, \ldots, 2g, \quad \forall \theta \in [-\delta, 0].$$

The considered function $v$ is a convex norm, but it is not continuously differentiable. For such a function, it suffices to consider the one sided derivative in the direction of the vector field $\dot{y}(t) = A_0 y(t) + \sum_{i=1}^{m} A_i y(t - \tau_i)$:

$$\frac{dv}{dt^+} = \lim_{\epsilon \to 0} \max_{\gamma(t) = \xi} [v(y(t + \epsilon)) - v(y(t))].$$

Here, one gets

$$\frac{dv}{dt^+} = \max_{\gamma(t) = \xi} \left[ I A_0 y(t) + \sum_{i=1}^{m} I A_i y(t - \tau_i) \right].$$

Positive invariance of $S(G, 1_g)$ can be characterized as in the proof of Proposition 1, by the conditions (16)--(18) with the vector $\gamma$ replaced by $1_{2g}$. Hence it follows that

$$\frac{dv}{dt^+} = \max_{\gamma(t) = \xi} \left[ H_0 J y(t) + \sum_{i=1}^{m} H_i J y(t - \tau_i) \right].$$

and it follows from (33) that

$$\frac{dv}{dt^+} = \max_{\gamma(t) = \xi} \left[ H_0 \cdots H_m \right] \xi 1_{2g(p+1)}.$$

From the proof of Proposition 1, the relation (31) is equivalent to $[H_0 \cdots H_m]1_{2g(p+1)} < 0_{2g}$. Then it follows that $\frac{dv}{dt^+} < 0$.

Thus the system (29) is asymptotically stable. As the relations (30) and (31) are independent of the delay values, asymptotic stability is obtained for any non-negative values of $\tau_1, \ldots, \tau_m$ in $[0, t]$.

In the case of the perturbed system (1), positive invariance of $S(G, \mu)$, with rank $(G) = n$, is obtained jointly with the asymptotic stability of the system (29) if the right-hand side of the inequality (31) is replaced by $-\omega 1_g$. Then $\omega$ can be interpreted as a robust stability margin for the system (29).
3.2. Comparison with other stability conditions

Some classical delay-independent stability conditions for multiple delay differential systems are established in Lewis and Anderson (1980), Hale et al. (1985) and Wang et al. (1987).

Wang et al. (1987) use the matrix measure of the matrix $A_0$, denoted by $v(A_0)$ and defined as follows:

$$v(A_0) = \lim_{\varepsilon \to 0} \frac{||I + \varepsilon A_0|| - 1}{\varepsilon}. \quad (34)$$

The matrix measure $v(A_0)$ is negative if $A_0$ is asymptotically stable. Then the stability condition given in Wang et al. (1987) for multiple delay systems is

$$\sum_{i=1}^{n} ||A_i|| < -v(A_0). \quad (35)$$

The condition (35) and the expression (34) are valid for any induced matrix norm $||\cdot||$. Given the choice of the infinity norm as the induced matrix norm, a condition less restrictive than (35) can be obtained by selecting a particular domain to be positively invariant. It suffices to set $G = I$ in Theorem 1 to obtain

$$(\hat{A}_0 + \sum_{i=1}^{n} |A_i|)I_g < 0. \quad (36)$$

The condition (36) can be interpreted as the positive invariance condition of the particular domain $S(I_0, 1_n)$ with respect to the system (29).

In the particular case $g = 1$, that is when (29) is scalar, the condition (36) reduces to $\sum_{i=1}^{n} |A_i| < -A_0$. This implies in particular that $\sum_{i=1}^{n} A_i < 0$ and $\sum_{i=1}^{n} |A_i| \leq |A_0|$, which are the stability conditions given in Hale et al. (1985) for the scalar linear case. These authors have also shown that less restrictive stability conditions for scalar delay systems may be established when the delays are linearly dependent.

In the case of uncertainties in the matrices $A_i, i = 0, \ldots, n$, the system (29) can be described by

$$\dot{y}(t) = (A_0 + \Delta A_0)y(t) + \sum_{i=1}^{n} (A_i + \Delta A_i)y(t - \tau_i) + Bu(t) + Ew(t). \quad (37)$$

Application of (35) in this case gives the robust stability conditions

$$\sum_{i=0}^{n} ||A_i|| < -v(A_0) - \sum_{i=1}^{n} ||A_i||. \quad (38)$$

For the infinity norm, a condition less restrictive than (38) is obtained from the positive-invariance approach under the choice $G = I_n$:

$$\sum_{i=0}^{n} ||A_i||_g < -\hat{A}_0I_g - \sum_{i=1}^{n} |A_i|_g \quad (39)$$

The general result for robust stability obtained from Theorem 1 allows a free choice of the matrix $G$ with rank $(G) = n$ and such that

$$\sum_{i=0}^{n} ||\Delta H_i||_g < -\hat{H}_0I_g - \sum_{i=1}^{n} |H_i|_g. \quad (40)$$

with

$$G(\Delta A_i) = (\Delta H_i)G, \quad i = 0, \ldots, p.$$ 

Positive invariance conditions for $S(G, \mu)$ with respect to the system (1) have been given in Proposition 1. These conditions can also be customized to the compact polytope $S(I_n, \mu)$, with $\mu$ a positive vector in $\mathbb{R}^n$. A sufficient delay-independent stability condition for the system (29) can then be derived: $\exists \mu_i, i = 1, \ldots, n, \sigma_i, \epsilon_i, \gamma_i$ such that $G = I_n$ and

$$\sum_{i=0}^{n} \mu_i(\hat{A}_0 + \sum_{k=0}^{p} |A_{i_k}|) + \sum_{i=0}^{n} \mu_i \sum_{k=0}^{p} |A_{i_k}| < 0. \quad (41)$$

These conditions generalize the ‘quasi-diagonal dominance conditions’ proposed, for a particular structure of delay terms, in Lewis and Anderson (1980).

4. INVARIANT CONTROL OF A DELAY INPUT-OUTPUT DIFFERENCE SYSTEM

4.1. The control design problem

The main result of the preceding section will now be used as the design objective for controlling the following delay differential model:

$$\dot{y}(t) = A_0y(t) + \sum_{i=1}^{n} A_iy(t - \tau_i) + Bu(t) + Ew(t). \quad (42)$$

where $y(t) \in \mathbb{R}^n$ is the instantaneous state vector, $u(t) \in \mathbb{R}^m$ the control vector and $w(t) \in \mathbb{R}^p$ a bounded external disturbance vector. The system delays $\tau_i, i = 1, \ldots, n$ are strictly positive. For the model (42) to be well defined, it is necessary to provide continuous and bounded initial conditions for $y(t)$ on $[t_0 - T, t_0]$, for the autonomous model (1).

Assuming now that the delays are known exactly, the investigated class of controls is described by

$$u(t) = K_0y(t) + \sum_{i=1}^{n} K_iy(t - \tau_i) \quad \text{for } t \geq t_0. \quad (43)$$

In contrast to the classical memoryless schemes, such as these derived from the quadratic stabilization approach (see e.g. Cheres et al., 1999; Shen et al., 1991; Klai et al., 1994; Niculescu et al., 1994), the proposed control law uses the information available about the delay terms.
For $t \geq t_0$, the closed-loop system takes the form

$$\dot{y}(t) = \tilde{A}_0 y(t) + \sum_{i=0}^{p} \tilde{A}_i y(t - \tau_i) + E\omega(t), \quad (44)$$

with

$$\tilde{A}_i = A_i + BK_i \quad \text{for} \quad i = 0, \ldots, p. \quad (45)$$

Suppose that the system instantaneous state is subject to symmetrical linear constraints

$$-1 \leq G y(t) \leq 1, \quad \forall t \geq t_0, \quad \text{with} \ G \in \mathbb{R}^{n \times n}. \quad (46)$$

Assuming that these constraints are satisfied for $t \leq t_0$, $\|G\omega(t)\| \leq \omega$ and rank $(G) = n$, the control objective can be described by satisfying constraints along the trajectories of the system $(42)$. This would imply robust asymptotic stability of the associated deterministic system

$$\dot{y}(t) = A_0 y(t) + \sum_{i=0}^{p} A_i y(t - \tau_i). \quad (47)$$

From Theorem 1, this problem can be solved by constructing, if possible, a sequence of gain matrices $K_i$, for $i = 0, \ldots, p$, and an associated sequence of $p + 2$ matrices $(\tilde{H}_i \in \mathbb{R}^{n \times n})$, for $i = 0, \ldots, p + 1$, such that

$$\tilde{H}_0 G = G A_0, \quad (48)$$

$$\tilde{H}_{p+1} G = G E, \quad (49)$$

$$\left( \tilde{H}_0 + \sum_{i=1}^{p} |\tilde{H}_i| \right) 1_g \leq -|\tilde{H}_{p+1}| \|\omega\|_g. \quad (50)$$

Several methods can be used, the matrix $G$ being given or having to be constructed. These methods can be based on those used for solving positive invariance relations for non-delay systems, particularly linear programming (Vassiliaki and Bitsoris, 1989; Döbrö and Milan, 1995).

From Theorem 1, positive invariance of $S(G, 1_g)$ with respect to the system $(44)$ is satisfied under external perturbations $\omega(t)$ and uncertainties $\Delta A_i$, as long as the following condition is satisfied;

$$-\delta + \sum_{i=0}^{p} |\Delta \tilde{H}_i| \leq -|\tilde{H}_{p+1}| \|\omega\|_g, \quad (51)$$

with $\Delta \tilde{H}_i$ defined by $(\Delta \tilde{H}_i)G = G(\Delta A_i)$, and $\delta$ a positive vector in $\mathbb{R}^n$ such that, for $j = 1, \ldots, g$,

$$-\delta_j = H_{0j} + \sum_{k \neq j} |H_{0jk}| + \sum_{i=1}^{p} \sum_{k=1}^{g} |H_{ijk}|. \quad (52)$$

The condition $(51)$ is a robust stability condition that also allows for the presence of limited uncertainties in the delay values $\tau_i$. If the delays are uncertain under the form $\tau_i + \Delta \tau_i$, the controller system $(44)$ takes the form

$$\dot{y}(t) = A_0 y(t) + \sum_{i=0}^{p} A_i y(t - \tau_i)$$

$$+ \sum_{i=1}^{p} BK_i [y(t - \Delta \tau_i - \tau_i) - y(t - \tau_i)] + E\omega(t).$$

Assuming that

$$\sum_{i=0}^{p} GBK_i [y(t - \Delta \tau_i - \tau_i) - y(t - \tau_i)] + G\omega(t) \leq \omega, \quad (53)$$

stability and positive invariance of $S(G, 1_g)$ are maintained under the condition

$$-\delta + (\sum_{i=0}^{p} |\Delta \tilde{H}_i|) 1_g \leq -\omega 1_g, \quad (54)$$

The condition $(53)$ can then be interpreted as a constraint on the gain matrices $K_i$.

4.2. Example

In this example, the basic data are taken from Shen et al. (1991). The model is as follows:

$$\dot{y}(t) = (A_0 + \Delta A_0) y(t) + (A_1 + \Delta A_1) y(t - \tau_1)$$

$$+ (B + \Delta B) u(t) + E\omega(t), \quad (55)$$

with $\|E\omega(t)\| \leq 0.5$. The delay $\tau_1$ is assumed to be known. The matrices of the system are defined by

$$A_0 = \begin{bmatrix} -3 & 2 \\ 19 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

and the perturbation matrices are

$$\Delta A_0 = \begin{bmatrix} \tau_1(t) & 0 \\ 0 & \tau_2(t) \end{bmatrix}, \quad \Delta A_1 = \begin{bmatrix} s(t) & s(t) \\ 0 & 0 \end{bmatrix}, \quad \Delta B = \begin{bmatrix} q(t) \\ 0 \end{bmatrix},$$

with $|\tau_1(t)| \leq 3, \ |\tau_2(t)| \leq 3, \ |s(t)| \leq 1$ and $|q(t)| \leq 0.1$.

The investigated class of controls takes the form

$$u(t) = K_0 y(t) + K_1 y(t - \tau_1) \quad \text{for} \ t \geq t_0 \quad (56)$$

It is not difficult to select the gain matrices $K_0$ and $K_1$ to guarantee positive invariance of the unit polyhedron $S(\tilde{H}_1, 1_g)$ with respect to the perturbed system $(55)$. For example, by LP minimization of a weighted sum of $\max H_{0j} (-\delta_j)$ and $(\|K_0\|_g + \|K_1\|_g$ under the relations

$$\tilde{H}_0 = \tilde{A}_0 = A_0 + BK_0, \quad \tilde{H}_1 = \tilde{A}_1 = A_1 + BK_1,$$
the following solution is obtained:

\[ K_0 = \begin{bmatrix} -4.75 & -2.00 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0.5 & 0 \end{bmatrix}, \]

\[ \bar{A}_0 = \begin{bmatrix} -7.75 & 0 \\ 0 & -10.00 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} -0.50 & -0.25 \\ 3.00 & 0 \end{bmatrix}. \]

Then \( \delta = (\bar{A}_0 + |\bar{A}_1|) l_2 = \begin{bmatrix} -7 \\ 7 \end{bmatrix} \). The vector of uncertainties is bounded by

\[ |\Delta A_0| + |\Delta A_1| + |\Delta B| (|K_0| + |K_1|) l_x, \]

and thus by \( \pi = \begin{bmatrix} 5.725 \\ 3.00 \end{bmatrix} \). It follows that

\[ \max_{i=1,2} (-\delta_i + \pi_i) = \max(-1.275, -4) = -1.275 \]

The value of the right-hand side, -1.275, is smaller than \( -\omega = -0.5 \). Therefore positive invariance of \( S(l_2, l_2) \) with respect to the closed-loop system and asymptotic stability of the associated deterministic system are guaranteed from the condition (54).

5. CONCLUSIONS

This paper has addressed the issue of robust stability for a perturbed delay system represented by a delay differential equation with the instantaneous state vector subject to linear symmetrical constraints and an additive bounded external disturbance vector.

The conditions guaranteeing asymptotic stability and the respect of linear constraints on the instantaneous state vector have been determined and used in the control design problem. The proposed methodology also applies when system parameters and delays are imperfectly known.

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