INvariance AND StABILITY BY STATE FEEDBACK FOR CONSTRAINED LINEAR SYSTEMS

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1 Problem Presentation

Consider the discrete-time linear dynamical system defined by the state equation (1):

\[ x_{k+1} = Ax_k + Bu_k \text{ for } k \in \mathbb{N} \]  

(1)

\[ x_k \in \mathbb{R}^n, \ u_k \in \mathbb{R}^m, \ A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m}, \ m \leq n. \]

The state vector \( x_k \) is subject to symmetrical linear constraints:

\[ -\rho \leq Gx_k \leq \rho \text{ for } k \in \mathbb{N} \]  

(2)

with \( G \in \mathbb{R}^{r \times n}, \ r \leq n, \ \text{rank}(G) = r, \ \rho \in \mathbb{R}^r \) and \( \rho_i > 0 \) for \( i = 1, \ldots, r \). These constraints are supposed to be satisfied by the initial state of the system, \( x_0 \).

The addressed problem is to find a closed-loop linear regulation law:

\[ u_k = Fx_k \]  

(3)

driving the state vector to zero while maintaining it in the symmetrical polyhedral domain \( R(G, \rho) \) defined by:

\[ R(G, \rho) = \{ x \in \mathbb{R}^n : -\rho \leq Gx \leq \rho \} \]  

(4)

A possible design principle is described in [13]. It consists of constructing a gain matrix \( F \) so as to stabilize the system and make \( R(G, \rho) \) a positively invariant set of the closed-loop system:

\[ x_{k+1} = A_0x_k \text{ with } A_0 = A + BF \]  

(5)

The following proposition can be used for characterizing the invariance of \( R(G, \rho) \). It has now become classical ([2], [3]), but it should be recalled as a basic result.

Proposition 1.1
The convex symmetrical polyhedral set \( R(G, \rho) \) is positively invariant for system (5) if and only if there exists a matrix \( H \in \mathbb{R}^{r \times r} \) such that:

\[ HG = GA_0 \]  

(6)

\[ |H|\rho \leq \rho \]  

(7)

By definition, \( |H| \) is the matrix of the absolute values of the components of matrix \( H \).

This result is a straightforward application to symmetrical polyhedral sets of the general invariance relationships for general polyhedral domains.

Set \( T = \begin{pmatrix} G & -G \\ -G & -G \end{pmatrix} \) and \( \nu = \begin{pmatrix} \rho \\ \rho \end{pmatrix} \). Invariance of \( R(G, \rho) \) is equivalent to:

\[ T(A + BF)x \leq \nu \text{ for any } x \in \mathbb{R}^n \text{ such that } Tx \leq \nu \]

By application of the extended Farkas’ lemma [4], the condition above is equivalent to the existence of a matrix of positive coefficients, \( K \in \mathbb{R}^{2r \times 2r} \), such that

\[ KT = T(A + BF) ; \ K\nu \leq \nu \]

When the condition of existence is satisfied, matrix \( K \) can as well be selected under the following form [3]:

\[ K = \begin{pmatrix} H^+ & H^- \\ H^- & H^+ \end{pmatrix} \]  

with \( (H^+)_{ij} \geq 0 \) \( \forall i, j = 1, \ldots, r \), \( (H^-)_{ij} \geq 0 \) \( \forall i, j = 1, \ldots, r \).

Clearly, \( H \) can be defined by \( H = H^+ - H^- \), and the existence of such a matrix \( K \) is equivalent to the existence of a matrix \( H \) satisfying relations (6), (7).

Conditions of proposition 1.1 cannot be satisfied by any system (5) and for any domain \( R(G, \rho) \). The main purpose of this study is to analyze under what "geometrical" properties of the triplet \( (A, B, G) \), there exists a positive vector \( \rho \) such that invariance of \( R(G, \rho) \) and stability can be obtained by state feedback.

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2 Positive invariance and-eigenstructure assignment

Relations (6) and (7) can be interpreted geometrically, in terms of (A,B)-invariance and stability, leading to lemma 2.1 and to proposition 2.2.

Lemma 2.1
A necessary and sufficient condition for the existence of a matrix \( H \in \mathbb{R}^{n \times r} \) satisfying relation (6) is that \( \text{Ker} \ G \) is an \((A,B)\)-invariant subspace of system (5).

Proof:
(a) necessity
Equation (6) can be equivalently replaced by:
\[ GA_0x = HGx \quad \forall x \in \mathbb{R}^n. \]
In particular, if \( x \in \text{Ker} \ G \), then \( GA_0x = 0 \), and this means \( A_0x \in \text{Ker} \ G \).
Necessarily, \( \text{Ker} \ G \) should be an \((A,B)\)-invariant subspace of \( \mathbb{R}^n \) and the gain matrix \( F \) should be a friend of \( \text{Ker} \ G \) ([14], p.88), so that, in geometric notations:
\[ (A + BF)\text{Ker} \ G \subset \text{Ker} \ G \] (8)

(b) sufficiency
If \( \text{Ker} \ G \) is \((A,B)\)-invariant, then, there exists a matrix \( F \in \mathbb{R}^{m \times n} \) such that:
\[ Gx = 0 \implies (A + BF)x = 0 \] (9)
Then,
\[ \text{Ker} \ G \subset \text{Ker} \ G(A + BF) \] (10)
From a well-known result of linear algebra, the orthogonal complementary subspace of \( \text{Ker} \ G \) in \( \mathbb{R}^n \) is spanned by the column-vectors of \( G^T \) and the orthogonal complement of \( \text{Ker} \ G(A + BF) \) in \( \mathbb{R}^n \) is spanned by the column-vectors of \( [G(A + BF)]^T \). Condition (10) implies:
\[ \text{Range}([G(A + BF)]^T) \subset \text{Range}(G^T) \] (11)
Therefore, relation (11) implies that each row-vector of \( G(A + BF) \) can be written as a linear combination of the row-vectors of \( G \), that is:
\[ (A + BF)_i = [H_1 ... H_r]G \quad \text{for} \ i = 1, ..., r \]
or
\[ G(A + BF) = HG \quad \text{with} \ H \in \mathbb{R}^{r \times r}. \]

The existence of such a vector, \( \rho \), can be derived from the existence of a control matrix \( F \in \mathbb{R}^{m \times n} \) such that:

(a) \( \text{Ker} \ G \) is an invariant subspace of system (5).

The real generalized eigenvectors of the restriction \((A + BF)\text{Ker} \ G\) are the column-vectors of matrix \( V_1 \) defined by:
\[ GV_1 = 0_{r \times (n-r)} \] (12)

(b) The real generalized eigenvectors associated to the eigenvalues of the restriction \((A + BF)(\mathbb{R}^n / \text{Ker} \ G)\) span a subspace \( L \subset \mathbb{R}^n \) such that \( L \oplus \text{Ker} \ G = \mathbb{R}^n \). They can be selected as the column-vectors of a matrix \( V_2 \) satisfying:
\[ GV_2 = I_{r \times r} \] (13)

The corresponding eigenvalues, \( \mu_i + j\sigma_i \), satisfy:
\[ |\mu_i| + |\sigma_i| < 1 \] (14)

Proof
We can solve the eigenstructure assignment problem by the same decomposition technique as in [14], proposition 4.1 pp.88-89. Let \( F_0 \) be a friend of \( \text{Ker} \ G \). The real generalized eigenvectors of \((A + BF_0)\text{Ker} \ G\) are the column-vectors of a matrix \( V_1 \) satisfying relation (12). The corresponding left generalized eigenvectors of \((A + BF_0)\text{Ker} \ G\) are the row-vectors of matrix \( G' \in \mathbb{R}^{(n-r) \times n} \) defined by:
\[ G'V_1 = I_{n-r \times n-r} \] (15)
Any feedback gain matrix \( F = F_0 + F_1G \), with \( F_1 \in \mathbb{R}^{m \times r} \) is also a friend of \( \text{Ker} \ G \). And \((A + BF)(\text{Ker} \ G)\) is identical to \((A + BF_0)(\text{Ker} \ G)\).
Consider the real Jordan canonical form of \((A + BF)\):
\[ J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \] (16)

\( J_1 \) is the real Jordan canonical form of \((A + BF)(\text{Ker} \ G)\). Matrix \( V_1 \) satisfies:
\[ G(A + BF)V_1 = J_1. \]
\( J_2 \) is the real Jordan canonical form of \((A + BF)(\mathbb{R}^n / \text{Ker} \ G)\). The restriction of \((A + BF_0)\) to \((\mathbb{R}^n / \text{Ker} \ G)\) is denoted \( A_0 \) and defined by the canonical projection equation:
\[ A_0G = G(A + BF_0). \]
Define \( \tilde{B} = GB \). Under the assumption that \((A,B)\) is controllable, then \((A_0, \tilde{B})\) is also controllable in \((\mathbb{R}^n / \text{Ker} \ G)\). The eigenvalues of \((A_0 + \tilde{B}F_1)\) can be selected so as to satisfy relations (14). Their associated generalized real Jordan form is matrix \( J_2 \).
Moreover, if \( \text{rank}(GB) = r \) and \( r \leq m \), we can select as the \( r \) generalized real eigenvectors which are not in \( \text{Ker} \ G \) the column-vectors of a matrix \( V_2 \) defined by relation (13). They span a subspace \( L \subset \mathbb{R}^n \) such that \( L \oplus \text{Ker} \ G = \mathbb{R}^n \).
To do so, it suffices to select \( \tilde{B} \) such that:
\[ BF_1 = J_2 - A_0 \] (17)
And in particular, \( F_1 = \tilde{B}^T(\tilde{B}\tilde{B}^T)^{-1}(J_2 - A_0) \) then, by construction, matrix \( V = [V_1 | V_2] \) is nonsingular. It satisfies:
\[ \begin{bmatrix} G' \\ G \end{bmatrix} V = V \begin{bmatrix} G' \\ G \end{bmatrix} = I_{n \times n} \] (18)
and \((A + BF)[V] = [V_1 | V_2] \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \). Or, equivalently, using relation (18),
\[ \begin{bmatrix} G' \\ G \end{bmatrix} \begin{bmatrix} A + BF \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} G' \\ G \end{bmatrix} \] (19)

Proposition 2.2
If the pair \((A,B)\) is controllable, \( \text{Ker} \ G \) an \((A,B)\)-invariant subspace, with \( \text{rank}(G) = r \leq m \), and \( \text{rank}(GB) = r \), there exists a positive vector \( \rho \) such that \( R(G, \rho) \) is positively invariant.
And in particular,
\[ G(A + BF) = J_2 G \]  
(20)

Relation (20) is equivalent to relation (6) when selecting \( J_2 \) as a candidate matrix \( H \). Under conditions (14) on the eigenvalues of \( J_2 \) (\( \lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_{n} \)) is an M-matrix [2]. Then, from a classical result ([9], [1]), this property implies the existence of a positive vector \( \rho \) such that relation (7) is satisfied for \( H = J_2 \).

\[ \square \]

3 Assignment of closed-loop poles to system zeros

As shown in the preceding section, the existence of a positive vector of \( \mathbb{R}^r, \rho \), for which \( R(G, \rho) \) is a positively invariant set of the closed-loop system (5) can be derived from the two following conditions:

1. Assignment of \( n - r \) generalized real eigenvectors in \( \text{Ker} \ G \).
2. Assignment of \( r \) real generalized eigenvalues in a complementary subspace of \( \text{Ker} \ G \), under conditions (14) on the eigenvalues of the restriction \( (A + BF)\mathbb{R}^n / \text{Ker} \ G \).

If we assume \((A,B)\) controllable and \( \text{rank}(GB) = r \leq m \), then, from proposition 2.2, the second requirement can be obtained from the first one.

The purpose of this section is to study under which geometric properties the system \( S(A,B,G) \) can meet these requirements. \( S(A,B,G) \) is defined by state equation (1) and by the output equation, with \( y_k \in \mathbb{R}^q \):

\[ y_k = Gx_k \text{ for } k \in \mathbb{N}. \]

The first condition for positive invariance of \( R(G,\rho) \) requires the existence of at least \( (n - r) \) (not necessarily distinct) zeros for system \( S(A,B,G) \). Then, condition 1 is fulfilled if \( n - r \) poles of the closed-loop system can be located at these zeros. The associated generalized real eigenvectors are the associated generalized real zero directions. They span \( \text{Ker} \ G \).

A similar eigenvectors assignment problem is treated in [12], but for continuous-time systems and with the purpose of reducing the model, rather than shaping its response. And actually, the placement of an eigenvalue at an invariant zero eliminates this mode (if it is stable) by "simplifying" the input-output map.

In a classical way, we can introduce the system matrix [11]:

\[ P(\lambda) = \begin{bmatrix} \lambda I - A & -B \\ G & 0_{r \times m} \end{bmatrix} \]

(22)

The zeros of \( S(A,B,G) \) are defined as the set of complex numbers \( \lambda_i \) for which there exist vectors \( v_i \in \mathbb{C}^n \) and \( w_i \in \mathbb{C}^m \) such that:

\[ P(\lambda_i) \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]  
(23)

\( v_i \) is called a state zero direction, and \( w_i \) an input zero direction.

Under the assumptions \( \text{rank}(G) = r, \text{rank}(B) = m \) and \( r \leq m \leq n \), two types of output zeros can be distinguished:

- **a**. The invariant zeros, defined as the set of complex numbers \( \lambda_i \) which make \( P(\lambda_i) \) rank deficient (see e.g. [12]), that is such that \( \text{rank}(P(\lambda_i)) < \min(n + r, n + m) \). From theorem 4.2 in [7], it is established that their associated zero directions generate the maximal \((A,B)\)-invariant subspace which has no intersection with \( B \). By definition, \( B \) is the image (or range) of \( B \).

- **b**. The "controllable" zeros, which can be located as desired. Their associated zero directions generate the maximal controllability subspace in \( \text{Ker} \ G \).

Let \( d \) be the rank deficiency of matrix \( GB \); \( \text{rank}(GB) = r - d \). To find the zeros of \( S(A,B,G) \), define respectively right and left annihilators of matrices \( G \) and \( B \), \( M \in \mathbb{R}^{n \times (n-r)} \) and \( N \in \mathbb{R}^{(m-r) \times n} \), satisfying the following relations: \( GM = 0_{r,n-r} \), \( NB = 0_{n,m-r} \). The degrees of freedom on the choice of matrices \( N \) and \( M \), and the property rank \((NM) = n - m - d \), allow to choose \( N \) and \( M \) so that [8]:

\[ NM = \begin{bmatrix} I_{n - m - d,n - m - d} \\ 0_{d,n - m - d} \end{bmatrix} \begin{bmatrix} 0_{n - m - d,m + d - r} \\ 0_{d,m + d - r} \end{bmatrix} \]

(24)

Any vector \( v_i \in \mathbb{C}^n \) satisfying relation (23) belongs to \( \text{Ker} G \subset \mathbb{C}^n \). Therefore, it is uniquely defined by the vector \( z_i \in \mathbb{C}^{n-r} \) such that:

\[ v_i = Mz_i \]  
(25)

Relation (23) can then be equivalently replaced by:

\[ [\lambda_i I - A - B] \begin{bmatrix} Mz_i \\ w_i \end{bmatrix} = 0 \]  
(26)

The \( m \) components of \( w_i \) can be eliminated by left multiplication of (26) by matrix \( N \), yielding:

\[ [\lambda_i NM - NAM]z_i = 0 \]  
(27)

Equations (23) and (27) have the same solutions \( \lambda_i \in \mathbb{C} \), which are the finite zeros of \( S(A,B,G) \). The polynomial matrix \( \lambda NM - NAM \) is called the zero pencil [8]. It completely characterizes the finite zeros and the associated zero directions of \( S(A,B,G) \).

Using for matrix \( NAM \) the same partitioning as for \( NM \), we can write the zero pencil as follows:

\[ \lambda NM - NAM = \begin{bmatrix} A - (NAM)_{1} & -(NAM)_{2} \\ -(NAM)_{3} & -(NAM)_{4} \end{bmatrix} \]  
(28)

3.1 The invariant zero directions

The rank deficiency of \( P(\lambda_i) \), \( d_i \), is the geometric multiplicity of \( \lambda_i \). The algebraic multiplicity of \( \lambda_i \) is \( m_i = \sum_{j=1}^{r} m_{ij} \). It is the power of the invariant factor \( \lambda - \lambda_i \) in the zero polynomial of \( S(A,B,G) \).

If \( m_i \) is strictly greater than \( d_i \), the invariant zero-directions and generalized invariant zero-directions associated to the string \( ij \) such that \( m_{ij} > 1 \) satisfy the sequence of equations [10]:

\[ P(\lambda_i) \begin{bmatrix} v'_{i,j} \\ w'_{i,j} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

And, for \( k = 2, \ldots, m_{ij} \),

\[ P(\lambda_i) \begin{bmatrix} v'_{i,jk} \\ w'_{i,jk} \end{bmatrix} = \begin{bmatrix} -v'_{i,j(k-1)} \\ 0 \end{bmatrix} \]
In this chapter, two cases will be considered: the case of regular pencils, corresponding to a non-degenerate square zero-pencil \((r = m)\), and the case of singular pencils.

### 3.1.1 Case \(r = m\) and regular zero pencil

If the zero pencil is square, and its determinant not always equal to 0 for any value of \(\lambda\), it is said to be regular and its kernel is uniquely determined by the set of elementary divisors of \(P(\lambda)\). In this regular case, the conditions of existence of a positively invariant controller (with or without global stability of the closed-loop system) can be stated as follows:

**Proposition 3.1**

If the zero pencil is regular, a necessary and sufficient condition to span \(\text{Ker} \, NAM\) with \(n - m\) independent generalized zero-directions is \(\text{rank}(GB) = n\) (matrix \(GB\) full-rank). Under this condition, it is always possible to obtain the positive invariance of \(R(G, \rho)\) for some positive vector \(\rho\). But stability of the closed-loop system is also obtained if and only if all the invariant zeros of \(S(A, B, C)\) are stable (located in the unit-circle of the complex plane).

**Proof**:

(a) case \(d = 0\).

If matrix \(GB\) is full-rank, that is for \(d = 0\), then, from (24), \(NAM = \mathbf{I}_{n-m,n-m}\). The \(n - m\) invariant zeros of \(\text{Ker} \, S(A, B, C)\) are finite. They are the eigenvalues of matrix \(NAM\).

To each finite invariant zero, \(\lambda_i\), with algebraic multiplicity \(m_i\), corresponds an \((A, B)\)-invariant subspace in \(\text{Ker} \, G\), with dimension \(m_i\). The direct sum of these elementary \((A, B)\)-invariant subspaces is \(\text{Ker} \, G\) itself. System zeros are invariant under any state feedback (see e.g. [8]).

If any of these zeros is located outside the unit circle of the complex plane, the requirement of locating closed-loop poles at invariant zeros would prevent from stabilizing the system. Positive invariance of \(R(\rho, G)\) and stability of the closed-loop system cannot both be obtained.

Note also that if \(GB\) is non-singular, \(B \cap \text{Ker} \, G = \{0\}\).

(b) case \(d > 0\).

If \(\text{rank}(GB) = m - d\) \((= r - d)\) with \(d > 0\), \(\text{rank}(NAM) = n - m - d\), the zero pencil has \"infinite elementary divisors\". The sum of the degrees of the associated invariant polynomials is \(d\).

Since the pencil is supposed to be regular, each zero direction corresponds to only one zero. The set of all the generalized zero directions associated to finite zeros spans an \((A, B)\)-invariant subspace included in \(\text{Ker} \, G\). This subspace is denoted \(V\) and has dimension \(n - m - d\).

The state zero directions associated to \"infinite zeros\" generate a subspace with dimension \(d\) having no intersection with \(V\). \"Infinite\" zero-directions are such that

\[
\begin{aligned}
NAMz &= 0 \\
NAM^2z &\neq 0
\end{aligned}
\]

For a regular pencil, condition \(\text{Ker} \, NAM \cap \text{Ker} \, NM = 0\) is always satisfied. Then, the subspace spanned by infinite zero-directions is precisely \(\text{Ker} \, G \cap B\). This subspace cannot be spanned by the solutions of (23) for finite values of \(\lambda_i\).

Then, clearly, the system \(S(A, B, G)\) only admits \(n - m - d\) finite invariant zeros and \(n - m - d\) associated zero directions defined by \(n - m - d\) independent vectors \((z_1, ..., z_{n-m-d})\) solutions of (27).

### 3.1.2 Cases of singular zero pencils

If \(r < m\) or if \(r = m\) but \(\text{det}(P(\lambda)) = 0\) \(\forall \lambda \in \mathbb{C}\) then \(P(\lambda)\) is said to be singular. In the last case, in order to have non-null solutions \(z_i\) to the zero pencil equation (27) for any value of \(\lambda_i\), we must necessarily have \(\text{dim}(\text{Ker} \, NM) = d > 0\). Note that in both cases, we have \(r < m + d\).

Then, invariant zeros and associated invariant zero directions are still associated to the elementary divisors of \(P(\lambda)\), if such divisors still exist. To each invariant zero direction is associated a unique value of \(\lambda \in \mathbb{C}\). The subspace of \(\text{Ker} \, G\) spanned by the invariant zero-directions is the maximal \((A, B)\)-invariant subspace of \(\text{Ker} \, G\) not intersecting \(B\). But some other solutions of equation (23) may exist. They are associated to the column-minimal indices of \(P(\lambda)\) and are denoted \"controllable zero-directions\".

As it will be shown in the next section, the choice of invariant zeros as closed-loop eigenvalues remains necessary. But zero-directions associated to a set of stable \"controllable\" zeros must be added to the set of invariant zero directions, and assigned as closed-loop eigenvectors to obtain the invariance of \(\text{Ker} \, G\).

From proposition 3.1, it is then clear that the following restrictions also apply to the singular case:

- If the system has invariant zeros and if any of these invariant zeros is infinite, invariance of \(\text{Ker} \, G\) will not be obtained.
- If all the invariant zeros are finite but if any of them is unstable, invariance of and global stability cannot both be obtained with any constant state feedback.

### 3.2 The \"controllable\" zero directions

The decomposition of the zero pencil given in (28) indicates that the zeros of system \(S(A, B, G)\) are also the zeros of the non-proper system

\[
[S(\text{NAM})_1, (\text{NAM})_2, -(\text{NAM})_3, -(\text{NAM})_4]
\]

Any zero-direction associated to a value of \(\lambda \in \mathbb{C}\) can be defined from a vector \(z \in C^{n-r}\) belonging to the transmission subspace \(\text{Tr}(\lambda)\) of system \((\text{NAM})_1, (\text{NAM})_2\).

Here, by extension of the classical definition of (state) transmission subspaces [6], we include the state components and the input components in the definition of transmission subspaces. \(\text{Tr}(\lambda)\) is defined as the kernel of the pole pencil: \([N_{n-m-d,n-m-d} - (\text{NAM})_1 | -(\text{NAM})_2]\).

Then, in order to satisfy relation (27) for some value of \(\lambda_i\), vector \(z_i\) has to satisfy the two following relations:

\[
[\lambda_i N_{n-m-d,n-m-d} - (\text{NAM})_1, -(\text{NAM})_2] z_i = 0
\]

\[
-(\text{NAM})_3 | -(\text{NAM})_4 z_i = 0
\]

We are now ready to state the following proposition:

**Proposition 3.2**

A necessary and sufficient condition for \(\text{Ker} \, G\) to be spanned by zero-directions is

\[
d = \dim(\text{Image}[-(\text{NAM})_3 | -(\text{NAM})_4]) = 0
\]

(or equivalently,

\[
d = \dim(\text{Ker}[-(\text{NAM})_3 | -(\text{NAM})_4]) = n - r
\]

This condition can be split into two alternatives:

- either \(d = 0\)

- either \(d = n - r\)
or $d > 0$ but $- (NAM)_3 = 0_{d,n-r}$ (as it is always the case, for instance, if $A = I_{n,n}$).

Then, the condition of $(A+BF)$-invariance of $\ker G$ can be satisfied but stability of the closed-loop system is also obtained if and only if all the invariant zeros of $S(A,B,C)$ are stable (located in the unit-circle of the complex plane).

Invariance and controllable zeros have to satisfy the two relations (29) and (30). But since the second condition does not depend on the value of $\lambda$, it constrains all the candidate vectors $z$ to belong to $\ker (NAM)_3 \setminus -(NAM)_4$. If this kernel is not the whole space $C^{n-r}$ but has dimension $n-r-\delta$ with $\delta > 0$, then the associated zero-directions $v = Mz$ can at most generate a subspace of $\ker G$ with dimension $n-r-\delta$.

**Sufficiency**

Let us now assume that the condition above ($\delta = 0$) is satisfied. We can then suppress condition (30), and concentrate on relation (29). As noted above, the considered singular pencils satisfy $r < m+d$. Then for any complex value of $\lambda$, the transmission subspace $\text{Tr}(\lambda)$ for system with state-matrix $((NAM)_1, (NAM)_2)$ has a dimension greater or equal to $m+d-r$.

$$\text{Tr}(\lambda) = \{z; z \in C^{n-r}; \lambda I - (NAM)_1 | -(NAM)_2 z = 0\}$$

(a) If $\dim[\text{Tr}(\lambda)] > m+d-r$ if $\lambda_i$ is an invariant zero of $S(A,B,G)$.

Note that the invariant zeros of $S(A,B,G)$, when they exist, are the "input decoupling" zeros of system $((NAM)_1, (NAM)_2)$ [8].

The system $((NAM)_1, (NAM)_2)$ has $s$ input decoupling zeros, these zeros are uncontrollable polynomials of $(NAM)_1$, and the maximal controllability subspace of the pair $((NAM)_1, (NAM)_2)$ has dimension $n-m-d-s$.

(b) If $\dim[\text{Tr}(\lambda)] = m+d-r$ if $\lambda_i$ is a "controllable" zero of $S(A,B,G)$, that is simply any complex value which is not an invariant zero.

A sufficient condition for the existence of $n-r$ zero-directions spanning $\ker G$ is that the union of all the transmission spaces $\text{Tr}(\lambda)$ (for at most $n-r$ values of $\lambda$) span $C^{n-r}$.

Any $z \in C^{n-r}$ can be written $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, with $z_1 \in C^{n-m-d}$ and $z_2 \in C^{m+d-r}$. If it belongs to $\text{Tr}(\lambda)$, it satisfies:

$$(NAM)_2 z_1 = \lambda z_1 - (NAM)_2 z_2$$

Under the assumption $\delta = 0$, the assignment of $n-r$ closed-loop eigenvectors in $\ker G$ can be obtained in two stages:

**Stage 1**

First, apply a state feedback $\Phi_1$ to $(NAM)_1, (NAM)_2$ to locate the $n-m-d-s$ eigenvalues of matrix $((NAM)_1 + (NAM)_2) \Phi_1$ in the stable region. These eigenvalues can be selected all different, and different from the uncontrollable poles (if there are any). In order for the set of vectors $z_1$ solutions of (31) to span $\mathbb{R}^{n-m-d}$, the set of closed-loop poles, $\lambda_i$, must include the uncontrollable poles of $(NAM)_1$ and $(NAM)_2$, with their order of multiplicity. As mentioned above, these uncontrollable poles are precisely the invariant zeros of $S(A,B,G)$.

The eigenvectors of $(NAM)_1$ define $n-m-d$ independent zero directions such that

$$\begin{cases} z_1 \neq 0 \\ z_2 = \Phi_1 z_1 \end{cases}$$

The corresponding zero-directions obtained from these vectors through relation (25) span a subspace $S_1$ in $\ker G$ with dimension $n-m-d$.

**Stage 2**

Any solution of equation (31) is also a solution of

$$(NAM)_2' z_1 = \lambda z_1 - (NAM)_2 z_2'$$

And under the change of coordinates in (32), the $n-r-d$ independent eigenvectors of $(NAM)_1'$ satisfy

$$\begin{cases} z_1 \neq 0 \\ z_2' = 0 \end{cases}$$

Consider the Jordan form associated to the controllable subspace of $(NAM)_1'$, $\Lambda = \Pi (NAM)_1' \Gamma$, where the ith line of $\Pi$ is the left-eigenvector of $(NAM)_1'$ and the ith column of $\Gamma$ the corresponding right eigenvector for eigenvalue $\lambda_i$, with all $\lambda_i$ distinct by construction for $i = 1, \ldots, m-d-s$.

Now, we can use a basis of $\mathbb{R}^{m+d-r}$ as $m+d-r$ independent input vectors, $z_2' \neq 0$, of $m+d-r$ different transmission subspaces $\text{Tr}(\mu_i)$ for any set of selected distinct eigenvalues $(\mu_1, \ldots, \mu_{m+d-r})$.

By construction, all the vectors $\begin{bmatrix} z_{1i} \\ z_{2i} \end{bmatrix}$ are independent and independent from vectors satisfying relation (33). The associated coordinates of vectors $z_i$ satisfying equation (31) are $\begin{bmatrix} z_{1i} \\ z_{2i} + \Phi_1 z_{1i} \end{bmatrix}$. The corresponding zero-directions obtained from these vectors through relation (25) span a subspace $S_2$ of $\ker G$ independent of $S_1$, with dimension $m+d-r$. Therefore, it is such that its direct sum with $S_1$ generates $\ker G$.

Note that the decomposition of $\ker G$ used in this proof is far from being unique. In fact, as it will be illustrated in the examples, we have a free choice of the $n-r-s$ controllable zeros. And the eigenstructure assignment problem can practically be solved in a single stage.

4 Implementation and Examples

To compute the eigenvectors of the closed-loop system, we use Moore's formulation of the pole pencil, $S(\lambda_i)$, and its kernel $K_{\lambda_i}$:

$$S(\lambda_i) = [\lambda I - A \mid -B] ; K_{\lambda_i} = \begin{bmatrix} N_{\lambda_i} \\ M_{\lambda_i} \end{bmatrix}$$

Then, any solution $v_i \in \mathbb{R}^n$ can be written, for some vector $k_i$ of appropriated dimension,

$$v_i = N_{\lambda_i} k_i, \quad \text{with} \quad F v_i = M_{\lambda_i} k_i$$

(34)
If the eigenvector belongs to $\text{Ker } G$, it should satisfy:

$$GN_{\lambda_i} k_i = 0$$  \hspace{1cm} (35)

and if it belongs to $\mathcal{L}$ such that $\mathcal{L} \oplus \text{Ker } G = \mathbb{R}^n$:

$$GN_{\lambda_i} k_i = e_i$$  \hspace{1cm} (36)

with $e_i$ the corresponding vector belonging to the canonical basis of $\mathbb{R}^n$.

These formulations can be used directly for simple real eigenvalues. In the case of simple complex conjugate eigenvalues, $(\mu_1 + j\sigma_1, \mu_2 - j\sigma_1)$, the complex kernels $(K_{\mu_1 + j\sigma_1}, K_{\mu_2 - j\sigma_1})$ are replaced by the associated real kernels of the real pencils defined as follows.

The closed-loop eigenvectors and controls associated to $\mu_1 + j\sigma_1$ and $\mu_2 - j\sigma_1$ (with $\sigma_i \neq 0$) are denoted:

$$\begin{bmatrix} v_{1i} \\ w_{1i} \end{bmatrix}, \begin{bmatrix} v_{2i}^* \\ w_{2i}^* \end{bmatrix}.$$  

They are complex conjugate. The corresponding real directions, denoted

$$\begin{bmatrix} v_{\mu_1} \\ w_{\mu_1} \\ v_{\sigma_1} \\ w_{\sigma_1} \end{bmatrix} = \begin{bmatrix} v_{1i} \\ w_{1i} \end{bmatrix}, \begin{bmatrix} v_{2i}^* \\ w_{2i}^* \end{bmatrix}$$

are defined as the real and imaginary parts of

$$\begin{bmatrix} v_{\sigma_1} \\ w_{\sigma_1} \end{bmatrix}$$

They satisfy the following equations:

$$\begin{bmatrix} \mu_1 I - A & -\sigma_1 I & -B \\ \sigma_1 I & \mu_2 I - A & 0 \end{bmatrix} \begin{bmatrix} v_{\mu_1} \\ v_{\sigma_1} \\ w_{\mu_1} \\ w_{\sigma_1} \\ v_{\sigma_1} \\ w_{\sigma_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  \hspace{1cm} (37)

Let $V = [V_1 \ V_2]$ be the matrix of the desired real generalized eigenvectors, and $W = [W_1 \ W_2]$ the associated control. The feedback gain matrix providing the desired eigenstructure assignment is:

$$F = WV^{-1}$$

### 4.1 Positive Invariance without global Stability

Consider the following data:

$$A = \begin{bmatrix} 0.4832 & 0.8807 & -0.7741 \\ -0.6135 & 0.6538 & -0.9626 \\ -0.2749 & 0.4899 & 0.9933 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.8360 & 0.4237 \\ 0.7469 & -0.2613 \\ -0.0378 & 0.2403 \end{bmatrix}$$

The open-loop system has unstable eigenvalues:

$$\lambda(A) = \begin{bmatrix} 0.4481 + j0.9683 \\ 0.4481 - j0.9683 \\ 1.2341 \end{bmatrix}$$

The state constraints are defined by

$$G = \begin{bmatrix} -0.5192 & 0.0189 & -0.3480 \\ -0.2779 & -0.7101 & -0.6953 \end{bmatrix}; \rho = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$$

System $S(A, B, G)$ has an unstable zero $\lambda_1 = 1.6166$. This value is selected as a closed-loop eigenvalue, for which the associated eigenvector spans $\text{Ker } G$. The 2 other poles have been chosen as follows:

$$\begin{bmatrix} \lambda'_1 = 0.1 + j0.6 \\ \lambda'_2 = 0.1 - j0.6 \end{bmatrix}.$$  \hspace{1cm} Then, we get

$$A_0 = \begin{bmatrix} 0.4618 & 1.0547 & -0.0583 \\ -0.4946 & 0.0478 & -1.4530 \\ -0.0875 & -0.3519 & 1.3070 \end{bmatrix}$$

On fig 1, we can see the stable time evolution of $x'_1$ and $x'_2$, which are the coordinates of the state vector in an orthonormal basis of the space spanned by the second and third columns of $V$. But the global unstability clearly stands out on fig 2. The third coordinate of the state vector has a divergent evolution.

In this example, global stability of the closed-loop system and positive invariance of $R(G, \rho)$ cannot be simultaneously achieved.

### 4.2 Positive Invariance with global Stability

Now, consider the same system as before, subject to constraints defined by:

$$G = \begin{bmatrix} -0.6538 & 0.7741 & -0.9933 \\ 0.4899 & -0.9626 & -0.8360 \end{bmatrix}; \rho = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$$

The two figures (3, 4) show that positive invariance of $R(G, \rho)$ and global asymptotic stability are obtained when selecting

$$F = \begin{bmatrix} -0.5117 & -0.0986 & -1.5022 \\ -0.2111 & -0.2378 & -3.3025 \end{bmatrix}$$

which yields: $A_0 = \begin{bmatrix} -0.0341 & 0.8624 & -0.9173 \\ -0.1761 & 0.6423 & -1.2219 \\ -0.3449 & 0.4365 & 0.2565 \end{bmatrix}$

and $V = \begin{bmatrix} -0.9129 & -0.8081 & -0.7202 \\ -0.0668 & -1.1403 & -0.4641 \\ -0.4580 & -0.3567 & 0.1123 \end{bmatrix}$

Under this choice, the poles of the closed-loop system are:

$$\begin{bmatrix} \lambda_1 = 0.6647 \hspace{1cm} (\text{stable zero}) \\ \lambda'_1 = 0.1 + j0.6 \\ \lambda'_2 = 0.1 - j0.6 \end{bmatrix}.$$  \hspace{1cm} Figure 1:
References


