(A,B)-INVARINCE CONDITIONS OF POLYHEDRAL DOMAINS FOR CONTINUOUS-TIME SYSTEMS

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Abstract

This paper provides an algebraic characterization of the (A,B)-invariance property of polyhedral sets with respect to linear continuous-time systems. The family of control laws which is investigated is the set of continuous and Lipschitz functions. Some particular conditions of existence of linear state feedback laws are also presented.

1 Introduction

The concept of (A,B)-invariance has shown to be very powerful when applied to subspaces, particularly for analyzing the existence of disturbance decoupling controls for linear systems [11]. This concept has been applied to polyhedral sets for discrete-time linear systems to characterize, through simple linear matrix relations, the possibility of controlling a system subject to hard trajectory constraints ([5], [6] [4] and references therein).

The problem of controlling continuous-time linear systems subject to linear constraints has been intensively studied in the last years, mainly in the framework of the positive invariance approach (see e.g. [10], [2] and references therein). A positively invariant domain in the state space is a domain from which the state vector trajectory cannot escape. An advantage of this approach is to provide an algebraic test for existence of a static state-feedback control law able to guarantee the respect of the constraints. In many cases however, closed-loop positive invariance cannot be obtained under a static state-feedback. One is then lead to consider more general control functions through the study of the (A,B)-invariance property. In this context, for compact polyhedra, a vertex-by-vertex characterization of the (A,B)-invariance property with respect to uncertain systems has been proposed in [1], through the study of a discrete-time approximation of the continuous system.

The main objective of this paper is to characterize the (A,B)-invariance property of general convex polyhedra with respect to continuous-time linear systems. Some preliminary results on cones and polyhedra are recalled in section 2. Then, explicit algebraic (A,B)-invariance conditions are obtained in section 3, from the application of Farkas’ Lemma. The (A,B)-invariance conditions take a particular form in the case of symmetrical polyhedra. A candidate control law is constructed, and, in section 4, conditions are provided under which a linear control law can guarantee positive invariance of the polyhedron with respect to the closed-loop system.

Notations and Definitions. In mathematical expressions, the symbol ";" stands for "such that". The components of a matrix M are noted M_{jk} and its row-vectors M_j.

By convention, inequalities between vectors and inequalities between matrices are componentwise. The absolute value |M| (resp. |v|) of a matrix M (resp. of a vector v) is defined as the matrix (resp. vector) of the absolute value of its components. An essentially nonnegative matrix M is a matrix having all its off-diagonal terms non-negative: M_{jk} \geq 0 \forall k \neq j. The cardinality of a set I, \text{card}(I), is defined as the number of elements in I.

2 Cones and Polyhedra

2.1 Polyhedral Sets

A polyhedron in \mathbb{R}^n, R[G, \rho], with G \in \mathbb{R}^{n \times n} and \rho \in \mathbb{R}^n, is defined by the system of linear inequalities:

R[G, \rho] = \{ x \in \mathbb{R}^n; Gx \leq \rho \}.

A polyhedral cone in \mathbb{R}^n, R[G, 0], is defined by the system of linear inequalities:

R[G, 0] = \{ x \in \mathbb{R}^n; Gx \leq 0 \}.

A symmetrical polyhedron in \mathbb{R}^n, S(Q, \phi), is defined, for \phi \geq 0, by the system of linear inequalities:

S(Q, \phi) = \{ x \in \mathbb{R}^n; |Qx| \leq \phi \}.
2.2 Generators of polyhedral cones

A set of generators of the polyhedral cone \( R[G, \rho] \) is defined as follows.

**Definition 1** The column-vectors of matrix \( M \) form a set of generators of the polyhedral cone \( R[G, \rho] \) if and only if there exists a non-negative vector \( \xi \) such that \( x = M \xi, \forall x \in R[G, \rho] \).

**Definition 2** A set of generators \( M \) of \( R[G, \rho] \) is called a minimal generating set if it has the smallest number of vectors.

Classically (see e.g. [8]), the affine hull (or lineality space) of \( R[G, 0] \) is \( \mathcal{A} = R[G, 0] \cap -R[G, 0] = \{ x \in \mathbb{F}^n; Gx = 0 \} \). The dimension of the affine hull is \( h = \text{rank}(G) \). In the general case, any polyhedral cone \( R[G, 0] \) can be decomposed into the form: \( R[G, 0] = P + \mathcal{A} \), where \( P \) is a proper cone [8]. If \( \mathcal{A} = \{ 0 \} \), the cone is pointed and a set of generators is obtained by selecting one non-zero vector of each extremal ray of the cone. This set of vectors forms a minimal generating set of \( R[G, 0] \) [3].

2.3 Decomposition of Polyhedra

Any polyhedron \( R[G, \rho] \subset \mathbb{F}^n \) admits a minimal decomposition (see e.g. [8]) as the sum of the cone \( R[G, 0] \), which is called its characteristic cone, and of a polytope, \( \Pi \), defined by its vertices \( (x_1, ..., x_p) \):

\[ \forall x \in R[G, \rho], \exists y \in R[G, 0], z \in \Pi ; x = y + z. \tag{1} \]

Each vertex \( (x_1, ..., x_p) \) of the polytope \( \Pi \) is a minimal face of the polyhedron \( R[G, \rho] \).

If the vectors \( M_j \ (j = 1, ..., q) \) form a set of generators of the polyhedral cone \( R[G, 0] \), then any point \( x \in R[G, \rho] \) can be defined by the set of parameters \( (\alpha_1, ..., \alpha_q) \) and \( (\beta_1, ..., \beta_p) \) through the linear expression:

\[ x = \sum_{j=1}^{q} \alpha_j M_j + \sum_{i=1}^{p} \beta_i x_i. \tag{2} \]

with \( \alpha_j \geq 0 \ \forall j = 1, ..., q; \ 0 \leq \beta_i \leq 1 \ \forall i = 1, ..., p; \sum_{i=1}^{p} \beta_i \leq 1. \)

Note that the polytope \( \Pi \) always belongs to the polyhedron \( R[G, \rho] \), as it can be shown by selecting \( y = 0 \) in decomposition (1). On the contrary, the characteristic cone \( R[G, 0] \) belongs to the polyhedron \( R[G, \rho] \) if and only if the zero vector belongs to \( R[G, \rho] \), that is, if all elements of vector \( \rho \) are non-negative. This condition will be assumed in the sequel.

For \( \rho \geq 0 \), a partition of \( R[G, \rho] \) can be derived from parameterization (2). This partition is an extension to the general case of the partition proposed by Gutman and Cwikel [5] and Blanchini [1] for compact convex polyhedra.

Each region \( \mathcal{X}_r \) of \( R[G, \rho] \) is generated through relation (2) by a set of generators and/or vertices \( (M_j, x_i), \ j \in J_r, \ i \in I_r \), such that:

- \( \text{card}(J_r) + \text{card}(I_r) = n. \)
- A point \( x \in \mathcal{X}_r \) is given by:
  \[ x = \sum_{j \in J_r} \alpha_j M_j + \sum_{i \in I_r} \beta_i x_i. \tag{3} \]

with \( \alpha_j \geq 0; 0 \leq \beta_i \leq 1; \sum_{i \in I_r} \beta_i \leq 1. \)

The transition between two adjacent regions is characterized by a pivoting operation in which one of the coefficients \( (\alpha_j, \beta_i) \) becomes null and either a generator \( M_j, j \in J_r \), or a vector \( x_i, i \notin I_r \), replaces, in representation (3), the generator or vector for which \( \alpha_j \) or \( \beta_i \) has become null. The intersection of two adjacent regions has an empty interior, and the union of all regions \( \mathcal{X}_r \) is the polyhedron \( R[G, \rho] \).

2.4 Farkas’ Lemma

The following form of Farkas’ Lemma will be used in this study (see e.g. [8]):

**Lemma 1** Let \( M \) be a matrix and \( v \) a vector. Then, \( \exists x : M x \leq v \text{ if and only if } y v \geq 0 \forall y \geq 0; y M = 0. \)

The set of candidate row-vectors \( y \geq 0 \) such that \( y M = 0 \) form a pointed polyhedral cone. In this study, this cone is called the non-negative left kernel of matrix \( M \).

Let \( W \) be a non-negative matrix whose row-vectors form a minimal generating set of the non-negative left kernel of \( M \). Then, a statement equivalent to Lemma 1 is (see also [3]):

\[ \exists x : M x \leq v \text{ if and only if } W v \geq 0. \tag{4} \]

As shown in [6], it is possible to compute matrix \( W \) by the Fourier-Motzkin elimination technique (see also [8]).

In the case of equality constraints, the following extended version of Farkas’ Lemma can be stated.

**Lemma 2** Let \( M \) and \( V \) be \( n \times m \) matrices of appropriate dimensions. Then, \( \exists x : M X = V \text{ if and only if } y V = 0 \forall y ; y M = 0. \)

3 (A,B)-invariance of polyhedra

Consider the linear system:

\[ \dot{x}(t) = Ax(t) + Bu(t), \tag{5} \]

with \( t \geq 0, x(t) \in \mathbb{F}^n, u(t) \in \mathbb{F}^m. \)

**Definition 3** A domain \( S \subset \mathbb{F}^m \) is said to be (A,B)-invariant (or controlled invariant) with respect to system (5) if for all initial state \( x(0) \in S \) there exists a control function \( u(t) \) such that the trajectory of the state vector \( x(t) \) remains in \( S. \)
Consider a polyhedral set of \( \mathbb{R}^n \), denoted \( R[G, \rho] \). (A,B)-invariance of such a polyhedral set for system (5) will now be characterized under the restriction that the control function \( u(x, t) \) belongs to the class denoted \( \mathcal{C}_L \) of functions which are continuous and Lipschitz with respect to the state vector. The following Proposition can then be shown, using similar arguments as in Seifert [9] (Corollary 1, p.295).

**Proposition 1** A necessary and sufficient condition for (A,B)-invariance of the set \( R[G, \rho] \) with respect to system (5), with a control function in the class \( \mathcal{C}_L \), is the existence of a control function in the class \( \mathcal{C}_L \) defined on the boundary of \( R[G, \rho] \), for which, at any point \( x \) of this boundary, the infinitesimal motion starting at \( x \) remains in \( R[G, \rho] \).

Consider now a matrix \( T \) whose row vectors form a minimal generating set of the non-negative left kernel of matrix \( GB \), denoted \( \Gamma \) and defined by:

\[
\Gamma = \{ w \in \mathbb{R}^n ; w^T(GB) = 0, \ w \geq 0 \}. \tag{6}
\]

Clearly this cone is pointed, because \( \Gamma \subset \mathbb{R}^n \). Therefore, as mentioned in section 2.2, a minimal generating set of \( \Gamma \) is obtained by selecting one non-zero vector of each extremal ray of \( \Gamma \).

**Proposition 2** A polyhedral set \( R[G, \rho] \subset \mathbb{R}^n \) is (A,B)-invariant with respect to system (5), with a control function in the class \( \mathcal{C}_L \), if and only if there exists a matrix \( Y \) having the same dimensions as \( T \) and such that:

\[
YG = TGA \tag{7}
\]
\[
Y \rho \leq 0 \tag{8}
\]
\[
Y_{ij} \geq 0 \text{ if } T_{ij} = 0. \tag{9}
\]

**Proof:**

**Necessity:** Assume (A,B)-invariance of \( R[G, \rho] \) with respect to system (5). Consider a row-vector \( T_{ij} \) of \( T \). From the definition of matrix \( T \), this vector is a generator of the cone \( \Gamma \). It has nonnegative components and satisfies:

\[
T_{ij}GB = 0. \tag{10}
\]

The following Linear Program, denoted \( P_i \), can be associated to this vector:

\[
\max_x \quad z_i = T_{ij}GAx
\]

subject to

\[
G_jx = \rho_j \quad \text{if } T_{ij} \neq 0 \tag{11}
\]
\[
G_jx \leq \rho_j \quad \text{if } T_{ij} = 0 \tag{12}
\]

The dual of problem \( P_i \), denoted \( D_i \), is defined as follows:

\[
\min_{y_i} \quad y_i = Y_{ij}\rho \tag{13}
\]

subject to

\[
Y_iG = T_{ij}GA \tag{14}
\]
\[
Y_{ij} \geq 0 \text{ if } T_{ij} = 0 \tag{15}
\]

- If the set of conditions (11), (12) is consistent, it defines a face of \( R[G, \rho] \). (A,B)-invariance of \( R[G, \rho] \) requires at each point \( x \) of this face admissibility of the infinitesimal motion, and thus the existence of a control vector \( u \) such that:

\[
G_jx = G_jAx + G_jBu \leq 0 \quad \forall j; \ T_{ij} \neq 0 \tag{16}
\]

Condition (10) implies \( \sum_j T_{ij}G_jB = 0 \). And since \( T_{ij} \geq 0 \), left-multiplication of each condition (15) by \( T_{ij} \) yields, for all \( x \) satisfying (11), (12):

\[
z_i = T_{ij}GAx = \sum_j T_{ij}G_jAx \leq 0.
\]

Thus, the optimal solution of problem \( P_i \), denoted \( y^*_i \), satisfies \( y^*_i = z^*_i \leq 0 \).

- If conditions (11), (12) are not consistent, the primal problem \( P_i \) is infeasible, but since \( R[G, \rho] \) is not empty, it can be made feasible by replacing some equality constraints \( G_jx = \rho_j \) by the associated inequality constraints \( G_jx \leq \rho_j \). A solution to the relaxed dual problem is then feasible for the original dual problem. Therefore, the optimal solution of the dual problem \( D_i \) is unbounded. Thus, there exists a row-vector \( Y_i \) satisfying (13), (14) and such that \( Y_i\rho \leq 0 \).

The same argument can be applied to all the row vectors of matrix \( T \), showing the necessity of conditions (7), (8), (9).

**Sufficiency:** Suppose the existence of a matrix \( Y \) satisfying conditions (7), (8), (9). Consider a point \( x \) on the boundary of \( R[G, \rho] \). The rows of matrix \( G \) and vector \( \rho \) can be partitioned into two subsets, with indices 1, 2, and dimensions \( g_1, g_2 \), and re-ordered as:

\[
G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad \rho = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix},
\]

so that this point satisfies:

\[
\begin{cases}
G_1x = \rho_1 \\
G_2x \leq \rho_2
\end{cases}
\tag{16}
\]

Consider the nonnegative matrix \( T_h \) whose row-vectors form a minimal generating set of the polyhedral cone \( \{ w \geq 0; w^T(G_1B) = 0 \} \). Any row vector \( T_{hi} \) can be complemented to generate a vector \( t_i \in \Gamma \) defined by:

\[
t_i = [T_{hi} 0].
\]

Then, by definition of matrix \( T \), \( \exists \xi_i \geq 0 \) such that \( t_i = \xi_i T \). Accordingly, all the row vectors of matrix \( t = [T_h 0] \) belong to \( \Gamma \), and thus, \( \exists \Xi \geq 0 \) such that:

\[
t = \Xi T
\]

Left-multiplication of relations (7), (8), by \( \Xi \) yields, with \( Z = \Xi Y \):

\[
ZG = tGA = T_hG_1A \tag{17}
\]
\[
Z\rho \leq 0. \tag{18}
\]
Furthermore, conditions \( \left\{ \begin{array}{l}
t_{ij} = \sum_k \xi_{ik} T_{kj} \\
z_{ij} = \sum_k \xi_{ik} y_{kj}
\end{array} \right. \) for each \( k \), respectively, \( \xi_{ik} = 0 \) or \( T_{kj} = 0 \). Using condition (9), this implies that, for each \( k \), respectively, \( \xi_{ik} = 0 \) or \( y_{kj} \geq 0 \), and thus,
\[
z_{ij} \geq 0 \text{ if } t_{ij} = 0 \tag{19}
\]
The three conditions (17), (18), (19) show, by duality, that, for \( x \) satisfying (16):
\[
T_0 G_1 Ax \leq 0 \tag{20}
\]
By application of Lemma 1 in the form (4), condition (20) is equivalent to the existence of a vector \( u \in \mathbb{R}^n \) such that
\[
G_1 \dot{x} = G_1 Ax + G_1 Bu \leq 0.
\]
This result applies to all the points on the boundary of \( R[G, \rho] \), in particular to all the vertices \((x_1, \ldots, x_p)\) of the polytope \( \Pi \). A set of admissible controls \((w_1, \ldots, w_p)\) can be associated to this set of vertices of the polytope. They satisfy:
\[
G_k A x_i + G_k B w_i \leq 0 \quad \forall k \; G_k A x_i = \rho_k.
\]
Similarly, the characteristic cone \( R[G, 0] \) trivially satisfy relations (7), (8), (9) with \( \rho = 0 \), \( Y = 0 \). Therefore, a set of controls \((w_1, \ldots, w_p)\) can be associated to a new set of generators \( M_j \) of \( R[G, 0] \), so as to satisfy:
\[
G_k A M_j + G_k B w_j \leq 0 \quad \forall k \; G_k M_j = 0.
\]
Each point of \( R[G, \rho] \) being represented by the set of coordinates \((\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p)\) through relation (2) of section 2.3, the following control function can be considered:
\[
u(x) = \sum_{j=1}^q \alpha_j w_j + \sum_{i=1}^p \beta_i v_i \tag{21}
\]
Using the set of coordinates derived from the partition of \( R[G, \rho] \) proposed in section 2.3 (with the assumption \( \rho \geq 0 \)), it can be shown that the control function (21) is defined at any point of \( R[G, \rho] \) is continuous and Lipschitz. It generates a feasible motion from each point of the boundary of \( R[G, \rho] \). Thus, using Proposition 1, the polyhedron \( R[G, \rho] \) is \((A, B)\)-invariant with respect to the system (5), with a control function in the class \( C_L \). \( \square \)

Remark:

- In the particular case when \( T \) reduces to the null row vector, the set of conditions (7)- (9) is trivially satisfied with \( Y \) equal to the null row-vector. The polyhedral set \( R[G, \rho] \) is then trivially \((A, B)\)-invariant.

- In the case of an autonomous systems \((B = 0)\), \( \Gamma \) (6) is the whole non-negative orthant \( \mathbb{R}^n_+ \). \( T = I_q \) and relations (7)-(9) reduce to the classical positive invariance relations for autonomous systems [10]. [2].

- The control law (21) can be seen as an extension to general polyhedra of the control law proposed in [5], [1] for compact polyhedra.

Proposition 2 can be specialized to the case of symmetrical polyhedra \( S(Q, \phi) \). Consider a matrix \([T_1 \; T_2] \) whose row vectors form a minimal generating set of the polyhedral cone \( \Gamma (6) \), with \( G = \left[ \begin{array}{c}
Q \\
-Q
\end{array} \right] \). Now, form the matrix \( T \) as a submatrix of \( T_1 - T_2 \), obtained by deleting the rows \( T_{1i} - T_{2i} \) for which either \( T_{1i} - T_{2i} = 0 \) or \( T_{1i} - T_{2i} = -T_{1j} + T_{2j} \) for some \( j < i \). The following result can be established. Its proof is omitted due to space limitation.

**Corollary** 1 A symmetrical polyhedral set \( S(Q, \phi) \subset \mathbb{R}^n \) is \((A, B)\)-invariant with respect to system (5) if and only if there exists a matrix \( Y \) having the same dimensions as \( T \) and such that:
\[
YQ = TQA, \tag{22}
\]
\[
\tilde{Y} \phi \leq 0, \tag{23}
\]
where \( \tilde{Y} \) is given by:
\[
\tilde{Y}_{ij} = \begin{cases} |Y_{ij}| & \text{if } T_{ij} = 0 \\ Y_{ij} & \text{if } T_{ij} > 0 \\ -Y_{ij} & \text{if } T_{ij} < 0 \end{cases}
\]

Very often, the desired property is not only \((A, B)\)-invariance, but also convergence to the origin with a prescribed rate. Consider then the function:
\[
\Psi(x) = \max_k \left\{ \frac{G_k x}{\rho_k} \right\} \tag{24}
\]
For compact polyhedra containing the origin in its interior, \( \Psi(x) \) is the Minkowski functional of \( R[G, \rho] \) [7]. It can be shown that in this case \( \Psi(x) \) is positive definite and continuous.

The total derivative of \( \Psi(x) \) with respect to system (5) is given by:
\[
D^+ (x, u) = \lim_{\Delta t \to 0^+} \sup \left\{ \frac{\Psi(x + \Delta t (Ax + Bu)) - \Psi(x)}{\Delta t} \right\}.
\]

**Definition** 4 A polyhedral compact domain \( R[G, \rho] \subset \mathbb{R}^n \) is said to be \((A, B)\)-invariant with an exponential convergence rate \( \epsilon > 0 \), if for all initial state \( x(0) \in R[G, \rho] \) there exists a control function \( u(t) \) such that the trajectory of the state vector verifies:
\[
D^+ (x, u) \leq -\epsilon \Psi(x).
\]

**Proposition** 3 A polyhedral compact domain \( R[G, \rho] \subset \mathbb{R}^n \) is \((A, B)\)-invariant with an exponential convergence rate \( \epsilon \), with a control function in the class \( C_L \), if and only if there exists a matrix \( Y \) such that:
\[
YG = TGA \tag{25}
\]
\[
Y \rho \leq -\epsilon T \rho \tag{26}
\]
\[
Y_{ij} \geq 0 \text{ if } T_{ij} = 0. \tag{27}
\]
Proof: The proof follows the same lines of the proof of Proposition 2 and is only outlined. Let \( I(x) \) denote the set of indices \( k \) for which \( \Psi(x) = \frac{G_k x}{\rho_k} \). By definition of \( \Psi(x) \),
\[
\{ \begin{array}{l}
G_k x = \rho_k \Psi(x) \quad \text{for} \quad k \in I(x) \\
G_k x < \rho_k \Psi(x) \quad \text{for} \quad l \notin I(x).
\end{array}
\]
Furthermore, by virtue of the continuity of \( \Psi(x) \), it can be shown that [10]:
\[
D^+(x,u) = \max_{k \in I(x)} \left\{ \frac{G_k(Ax + Bu)}{\rho_k} \right\}.
\]  
(28)

Now, consider the matrix \( T_b = [T_{b_k} 0] \), where the row vectors of \( T_{b_k} \) form a set of generators of the polyhedral cone \( \{ u \geq 0; \ w^T(G_k B) = 0 \} \). Then, from (25)-(27) and the proof of Proposition 2, there exists a matrix \( Y_b \) with \( Y_{b_{ij}} \geq 0 \) if \( T_{b_{ij}} = 0 \), such that \( T_b G_A x = Y_b G x \leq Y_b \rho \Psi(x) \). Therefore, from Lemma 1, there exists a control vector \( u \) such that \( G_k Ax + G_k Bu \leq -\varepsilon \rho_k \Psi(x) \) and thus, \( D^+(x,u) \leq -\varepsilon \Psi(x) \). \( \Box \)

Note that in this case \( \Psi(x) \) is a Lyapunov function of system (5).

4 Linear and piecewise linear control laws in a particular case

The \((A,B)\)-invariance results presented in section 3 assumes that the control is continuous and Lipschitz. In this framework, the control law (21) has been proposed. However, if the polyhedron \( R[G, \rho] \) has to be partitioned into many regions, the implementation of this law can become very difficult. It has been observed that in most cases closed-loop positive invariance can as well be obtained using much simpler state-feedback laws, which are simply linear. The linear state-feedback case can be characterized by \((A,B)\)-invariance conditions which are slightly stronger than those of Proposition 2.

Proposition 4 Let the row vectors of matrix \( T \) form a set of generators of the polyhedral cone \( \Gamma \) (6) and the row vectors of matrix \( [T \ N] \) span the left kernel of the map \( GB \). If there exists an essentially nonnegative matrix \( H \) such that
\[
[T \ N] \quad H G = [T \ N] \quad G_A,
\]  
(29)
\[
T H \rho \leq 0,
\]  
(30)
then the polyhedral set \( R[G, \rho] \subset \mathbb{R}^n \) is \((A,B)\)-invariant with respect to system (5) and positively invariant under a linear state feedback law:
\[
u = Fx + u_c.
\]  
(31)

Proof: \((A,B)\)-invariance of \( R[G, \rho] \) is easily shown by application of Proposition 2, with \( Y = TH \).

Now, consider matrix \( K = [T \ N] \), whose row-vectors span the left kernel of map \( GB \). The relation \( K H G = KGA \) with the row vectors of \( K \) spanning the left kernel of \( GB \) implies, from Lemma 2, the existence of a matrix \( F \) such that:
\[
HG = GA + GBF.
\]  
(32)

Application of Lemma 1, where matrix \( GB \) stands for matrix \( M \) and vector \( (\rho - \rho \) for \( v \), shows the equivalence of condition (30) with the existence of a constant vector \( u_c \) such that:
\[
H \rho + GBu_c \leq 0.
\]  
(33)

Using the power series representation of \( e^{(A+B)F \tau} \) and \( e^{H \tau} \), and (32), the following properties can be easily verified:
\[
Ge^{(A+B)F \tau}e^{H \tau} = e^{H \tau}G,
\]
\[
e^{H \tau} = I + \int_0^t e^{H \theta} d\theta
\]
\[
= \int_0^t e^{H(t-\theta)} dB u_c d\theta.
\]
Moreover, if \( H \) is essentially non-negative, then \( e^{H \tau} \geq 0 \). Therefore, the choice of a state feedback law (31) verifying (32), (33) guarantees, \( \forall \ x_0 = x(0) \in R[G, \rho], \)
\[
Gx(t) = Ge^{(A+B)F \tau}x_0 + \int_0^t Ge^{(A+B)F \tau} dB u_c d\theta = \]
\[
e^{H \tau}Gx_0 + \int_0^t e^{H(t-\theta)} dB u_c d\theta \leq \]
\[
e^{H \tau} \rho + \int_0^t e^{H(t-\theta)} dB u_c d\theta = \]
\[
(\int_0^t e^{H(t-\theta)} dB) \rho + \int_0^t e^{H(t-\theta)} dB u_c d\theta \leq \rho.
\]  
(34)

It can be shown that conditions (29), (30) are also necessary for existence of a linear state feedback law (31) for which \( R[G, \rho] \) is positively invariant.

If the constant control term, \( u_c \) is not null, the trajectory of the closed-loop system does not converge to the zero state. If it converges, it does to a state \( x_f \in R[G, \rho] \) which satisfies: \((A + BF)x_f = -Bu_c \).

Assuming \( R[G, \rho] \) compact, convergence to the origin can nevertheless be obtained under a variable structure control law. \( R[G, \rho] \) can be divided into \( g \) sectors, each of them being the convex hull of the origin and one facet. A vector \( x \in R[G, \rho] \) is included in the sector associated to the \( k \)-th facet, where \( k = \arg \max_{l} \frac{G_l x}{\rho_l} \).

Proposition 5 If \( R[G, \rho] \) is \((A,B)\)-invariant with an exponential convergence rate \( \varepsilon > 0 \) under a control law (31), then \( \Psi(x) \) (24) is a Lyapunov function of system (5) under the variable structure control law:
\[
u = Fx + u_c \Psi(x).
\]  
(34)
Proof: The total derivative of $\Psi(x)$ with respect to system (5) is given by (28):

$$D^+(x, u) = \max_{k \in I(x)} \left\{ \frac{G_k(Ax + Bu)}{\rho_k} \right\}$$

$$= \max_{k \in I(x)} \left\{ \frac{G_k((A + BF)x + Bu_c\Psi(x))}{\rho_k} \right\}$$

$$= \max_{k \in I(x)} \left\{ \frac{H_kGx + G_kBu_c\Psi(x)}{\rho_k} \right\} \text{ (from (32)).}$$

By definition of $\Psi(x)$, $G_kx = \rho_k\Psi(x)$ for $k \in I(x)$ and $G_lx < \rho_l\Psi(x)$ for $l \notin I(x)$.

Therefore, since $H$ is essentially non-negative,

$$D^+(x, u) \leq \max_{k \in I(x)} \left\{ \frac{(H_k\rho + G_kBu_c)\Psi(x)}{\rho_k} \right\} < 0$$

by virtue of (33), the hypothesis $\epsilon > 0$ and the positive definiteness of $\Psi(x)$. \qed

Note that in the interior of the $k$-th sector, the control law is a simple state-feedback: $u = (F + u_c\frac{G_k}{\rho_k})x$.

By virtue of the continuity of $\Psi(x)$, the control is uniquely determined in the boundary between two sectors $k$ and $l$ (note that the boundary is characterized by $\frac{G_kx}{\rho_k} = \frac{G lx}{\rho _l}$).

In [1] an alternative variable structure control law is proposed. This law is however more complex than the control law (34) because the number of sectors into which the polytope is divided is always greater than or equal to $p$.

In practice, the existence of $H \succeq 0$, $F$ and $u_c$ satisfying relations (32), (33) can be directly tested, and the solution constructed by Linear Programming [10].

5 Conclusion

(A,B)-invariance of general convex polyhedral sets with respect to continuous-time linear systems has been characterized by linear matrix relations which only depend on the system matrices and on the considered polyhedral set. These relations guarantee the existence of a continuous Lipschitz control law for which the polyhedron is positively invariant. In general, such a control law is not a linear state feedback. However, a set of conditions, slightly stronger than the (A,B)-invariance conditions, has been established to characterize the existence of a linear state feedback control law under which the (A,B)-invariant domain is positively invariant.

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References


