Applications of Controlled Invariance to the $l^1$ Optimal Control Problem

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Abstract

The purpose of this paper is to extend some results recently obtained on the $l^1$ optimal control problem using the controlled invariance approach. The major extension presented here concerns the case when the $l^\infty$ constraint on the output sequence defines an unbounded domain of the state space. An explicit characterization of the controlled invariance property is obtained. It is used to improve the existing algorithms for computing maximal controlled invariance sets, and to find in a straightforward way an $l^1$ optimal controller for a class of systems.

1 Introduction

Similarly to the $H_\infty$ optimal control problem, the $l^1$ optimal control problem seeks to minimize the influence of additive disturbances on the system outputs. In the $H_\infty$ control problem, this influence is measured in terms of the amplification of the disturbance input energy. In the $l^1$ control problem, this influence is measured in terms of the maximal possible amplification of the disturbance input magnitude.

A complete solution to the $l^1$ optimal control problem was given in [4]. Through the use of a parameterization of all stabilizing controllers, an optimal, possibly dynamic, linear controller is obtained by solving appropriate linear programs. In [13], it was shown that static non-linear controllers can provide the system with the same performance as linear dynamic ones. The same result was obtained in a constructive manner in [3]: a variable-structure controller was constructed by associating the achievement of an $l^1$ performance to the existence of a set of the state space which can be made invariant by state feedback.

The same idea was used in the paper [14]. But unlike [3] which relies on a linear dynamic controller to construct the static controller, it seeks to determine the best performance level for which there exists a corresponding controlled invariant set. A given performance level defines an admissible region in the state space. Such a performance can be achieved if and only if the largest controlled invariant set contained in the admissible region is not empty.

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This paper uses the controlled invariance approach to extend some existing results on the $l^1$ optimal control problem. An explicit characterization of the controlled invariance property is used to address the following problems:

- Construction of an efficient algorithm to compute the largest controlled invariant set contained in the admissible region defined by a given performance level.
- Solution of the $l^1$ optimal control problem for a class of minimal phase systems.

**Notations.** The set of real numbers is denoted $\mathbb{R}$ and the set of nonnegative integers is denoted $\mathbb{N}$. In mathematical expressions, the symbol ”;” stands for ”such that”. The components of a matrix $M$ are noted $M_{jk}$, and its row vectors $M_j$. By convention, inequalities between vectors and inequalities between matrices are componentwise. The absolute value $|M|$ (resp. $|v|$) of a matrix $M$ (resp. of a vector $v$) is defined as the matrix (resp. vector) of the absolute value of its components. Matrix $I_n$ denotes the identity matrix in $\mathbb{R}^{n \times n}$. Vector $1$ denotes a vector of appropriate dimension whose components are all equal to 1.

## 2 Problem Formulation

Classically, the $l^1$ optimal control problem is the problem of optimally limiting the maximal amplitude of the output of a system subject to persistent bounded disturbances.

As in [4], [6], [3], [14], the case of full state feedback is considered. The considered discrete time systems are described by:

\begin{align*}
    x_{k+1} &= Ax_k + B_1w_k + B_2u_k \\
    z_k &= Cx_k
\end{align*}

where $x_k \in \mathbb{R}^n$ is the state vector, $w_k \in \mathbb{R}^q$ the disturbance input vector, $u_k \in \mathbb{R}^m$ the control input vector and $z_k \in \mathbb{R}^p$ the controlled output vector.

The disturbance vector $w_k$ is supposed to be restricted to a bounded set $\mathcal{D}$:

\[ w_k \in \mathcal{D} \]

In the multivariable case, the maximal signal amplitudes of the disturbance and the controlled output vectors are usually measured in terms the $l^\infty$-induced norm, defined for bounded sequences $h = \{h_k\}$, by:

\[ \|h\|_{l^\infty} = \sup_{k \in \mathcal{N}} \|h_k\|_{\infty} < \infty. \]

The $l^\infty$ induced norm of system (1), (2) is equal to $\gamma$ if

\[ \sup_{\|w\|_{l^\infty} \leq 1} \|z\|_{l^\infty} = \gamma. \]

This implies for all $k \in \mathcal{N}$:

\[ |Cx_k| \leq \gamma 1; \quad \forall w = \{w_l\} ; \quad |w_l| \leq 1 \quad \forall l \in \mathcal{N} \]

Under this formulation, the domain of admissible disturbance inputs is:

\[ \mathcal{D} = S(I_q, 1) = \{w_k ; |w_k| \leq 1\} \]
and under an $l^1$ performance of $\gamma$, the domain of admissible state vectors is:

$$S(C, \gamma) = \{ x_k : |Cx_k| \leq \gamma \}. \quad (7)$$

As shown in [3] and in [14], the problems of existence and construction of a static state feedback achieving a given $l^1$ performance $\gamma$ rely on the property of controlled invariance for difference inclusions. This property can be analyzed either from viability theory [1] or from the geometric approach [15] (as in [8]).

### 3 Controlled invariance of polyhedral sets

In this paper, the definition of controlled invariance applies to general convex sets and with respect to differential inclusions. The term (A,B)-invariance [15] is used only for subspaces and with respect to deterministic systems.

**Definition 1** A domain $S \subset \mathbb{R}^n$ is controlled invariant with respect to system (1)-(3) if $\forall x \in S$ there exists an admissible control vector, $u \in \mathbb{R}^m$, such that: $Ax + B_1 w + B_2 u \in S$, $\forall w_k \in D$.

Note that this definition supposes that the disturbance vector is not measured.

The maximal one step admissible domain of a set $S$ is defined by [2]:

$$Q(S) = \{ x \in \mathbb{R}^n ; \exists u \in \mathbb{R}^m ; Ax + B_1 w + B_2 u \in S, \forall w \in D \}. \quad (8)$$

The controlled invariance property is thus equivalent to the geometric condition:

$$S \subset Q(S). \quad (9)$$

The framework of the $l^1$ optimal control problem can here be slightly extended by characterizing the conditions under which the sets $S, D$ are general convex polyhedral sets defined by:

$$S = R[G, \rho] = \{ x \in \mathbb{R}^n ; Gx \leq \rho \},$$

$$D = R[E, \mu] = \{ w \in \mathbb{R}^q ; Ew \leq \mu \}.$$

At any instant $k$, admissibility of the next state vector $x_{k+1}$ is characterized by the following constraints:

$$G A x_k + G B_2 u_k \leq \rho - \delta. \quad (10)$$

where

$$\delta_i = \max_{w \in R[E, \mu]} G_i B_1 w. \quad (11)$$

These constraints define a polyhedral set, $\Pi$, in the extended state space defined by the extended state vector $\begin{bmatrix} x_k \\ u_k \end{bmatrix}$. The maximal one step admissible state domain associated to (10), $Q(R[G, \rho])$, is the projection of the polyhedral set $\Pi$ onto the state space. To characterize this projection, the following form of Farkas Lemma will be used (see e.g. [12]).

**Lemma 1** Let $M$ be a matrix and $v$ a vector. Then, $\exists x ; Mx \leq v$ if and only if $yv \geq 0 \forall y \geq 0 : yM = 0.$


The set of candidate row-vectors \( y \) such that \( (y \geq 0, yM = 0) \) form a pointed polyhedral cone. In this study, this cone is called the nonnegative left kernel of matrix \( M \).

Let \( W \) be a nonnegative matrix whose row-vectors form a minimal generating set (see [5], [8]) of the nonnegative left kernel of \( M \). By convention, \( W \) is reduced to the null row-vector if the nonnegative left kernel only contains the null row-vector. In this case, the condition of Lemma 1 is trivially satisfied. Then, a statement equivalent to the statement in Lemma 1 is (see also [5]):

\[ \exists x ; Mx \leq v \text{ if and only if } Wv \geq 0. \] (12)

It is possible to compute matrix \( W \) by the Fourier-Motzkin elimination technique (see [12], [11]).

The set \( Q(R[G, \rho]) \) can be characterized as follows.

**Proposition 1** The maximal one step admissible domain \( Q(R[G, \rho]) \) is the polyhedral domain \( R[TGA, T(\rho - \delta)] \), where vector \( \delta \) is given by (11) and the row vectors of matrix \( T \) form a minimal generating set of the nonnegative left kernel of matrix \( GB \), denoted \( \Gamma \) defined by:

\[
\Gamma = \{ w \in \mathbb{R}^g; (GB_2)^T w = 0, w \geq 0 \}. \] (13)

**Proof:** From the definition (8) of \( Q(S) \), with \( S = R[G, \rho] \),

\[ Q(R[G, \rho]) = \{ x \in \mathbb{R}^n ; \exists u \in \mathbb{R}^m ; GAx + GB_2u \leq \rho - \delta \} \]

To show the proposition, it suffices to apply Lemma 1, under the form (12), with \( M = GB_2 \), \( v = \rho - \delta - GAx \). It gives the necessary and sufficient condition \( T(\rho - \delta - GAx) \geq 0 \), equivalent to \( x \in R[TGA, T(\rho - \delta)] \). ☐

Controlled invariance of the polyhedral set \( R[G, \rho] \) for system (1)-(3) with \( D = R[E, \mu] \) can then be characterized as in the following Proposition. The geometric condition (9) is translated into an algebraic condition using the extended Farkas Lemma (see [8] for the proof).

**Proposition 2** A polyhedral set \( R[G, \rho] \subset \mathbb{R}^n \) is controlled invariant with respect to system (1)-(3) if and only if there exists a nonnegative matrix \( Y \) such that

\[
YG = TGA \quad \text{and} \quad Y\rho \leq T(\rho - \delta) \] (14) (15)

where the row vectors of matrix \( T \) form a set of generators of the polyhedral cone \( \Gamma \) (13).

Proposition 2 can be specialized to the case of symmetrical polyhedra

\[ S(Q, \phi) := \{ x \in \mathbb{R}^n ; |Qx| \leq \phi \}, \]

\[ D = S(E, \mu) := \{ w \in \mathbb{R}^q ; |Ex| \leq \mu \}. \]

Consider a matrix \([T_1 \ T_2]\) whose row vectors form a minimal generating set of the polyhedral cone \( \Gamma \) (13), with \( G = \begin{bmatrix} Q \\ -Q \end{bmatrix} \).

Now, form the matrix \( T \) by deleting from matrix \( T_1 - T_2 \) the rows \( T_1i - T_2i \) for which \( T_1i - T_2i = 0 \) or \( T_1i - T_2i = -T_1j + T_2j \) for some \( j < i \).

The following result then follows.
Corollary 1 A symmetrical polyhedral set \( S(Q, \phi) \subset \mathbb{R}^n \) is controlled invariant with respect to system (1)-(3), with \( D = S(E, \mu) \), if and only if there exists a matrix \( Y \) such that :

\[
YQ = TQA,
\]

\[
|Y|\phi \leq |T|(|\phi - \delta|),
\]

where \( \delta_i = \max_{w \in \mathbb{R}[E,\mu]} Q_i B_1 w \).

**Proof**: From Proposition 2, \( S(Q, \phi) \) is controlled invariant if and only if there exist nonnegative matrices \( Y_1 \) and \( Y_2 \) such that :

\[
(Y_1 - Y_2)Q = (T_1 - T_2)QA,
\]

\[
(Y_1 + Y_2)\phi \leq (T_1 + T_2)(|\phi - \delta|).
\]

The rows \( i \) for which \( T_{1i} - T_{2i} = 0 \) do not need to be considered because in this case relations (18), (19) are trivially verified with \( Y_{1i} = Y_{2i} = 0 \). The same applies to the rows \( i \) for which \( T_{1i} - T_{2i} = -T_{1j} + T_{2j} \) for some \( j < i \), because if for the row \( j \) there exist row vectors \( Y_{1j} \) and \( Y_{2j} \) such that (18), (19) are verified, then the same relations are verified for the row \( i \) with \( Y_{1i} = Y_{2j} \) and \( Y_{2i} = Y_{1j} \).

Considering now the matrix \( T \), the following hold (otherwise the corresponding generators in \([T_1 \ T_2]\) would not belong to the minimal generating set) :

- \( T_{ij} = 0 \) only if the corresponding elements in \( T_1 \) and \( T_2 \) are also null,

- \( T_{ij} > 0 \) \((< 0)\) only if the corresponding elements in \( T_1 \) and \( T_2 \) are respectively positive \((\text{null}) \) and null \((\text{positive})\).

In view of these facts, from relations (18), (19), controlled invariance of \( S(Q, \phi) \) is equivalent to the existence of nonnegative matrices \( Y_1 \) and \( Y_2 \) verifying :

\[
(Y_1 - Y_2)Q = TQA,
\]

\[
(Y_1 + Y_2)\phi \leq |T|(|\phi - \delta|),
\]

Now, let \( Y = Y_1 - Y_2 \) and consider the matrices \( Y^+ \) and \( Y^- \) defined by :

\[
Y_{ij}^+ = \max\{Y_{ij}, 0\}
\]

\[
Y_{ij}^- = \max\{-Y_{ij}, 0\}.
\]

These matrices are such that \( Y^+ - Y^- = Y = Y_1 - Y_2 \) and \( (Y^+ + Y^-)\phi = |Y|\phi \leq (Y_1 - Y_2)\phi \leq |T|(|\phi - \delta|) \). The Proposition then follows from the fact that \( Y^+ \) and \( Y^- \) verify relations (20), (21). □

Note that any row vector \( t \) belonging to the left kernel of map \( QB \) can be written in the form \( t = t_1 - t_2; \ t_1, \ t_2 \geq 0 \). This means that \([t_1 \ t_2]^T\) belongs to \( \Gamma \) (13), with \( G = \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix} \). Therefore, matrix \( T \) necessarily contains as a sub-matrix a row-vector basis of the left kernel of map \( QB \).
4 The controlled invariance approach to the $l^1$-optimal problem

The relevance of the controlled invariance approach to the $l^1$-optimal problem will now be more precisely described.

4.1 The maximal controlled invariant domain contained in a polyhedron

Let $\mathcal{J}(R[G, \rho])$ represent the family of all controlled invariant polyhedra contained in $R[G, \rho]$. The maximal element of $\mathcal{J}(R[G, \rho])$ (when it exists) is defined as the member of $\mathcal{J}(R[G, \rho])$ which contains every member of $\mathcal{J}(R[G, \rho])$.

It can be verified that the convex hull of the union of two controlled invariant polyhedra is a controlled invariant polyhedron as well. Therefore, the family of all controlled invariant polyhedra is an upper semilattice relative to the operation “convex hull of the union”. This assures the existence of the set

$$C_\infty(R[G, \rho]) := \text{maximal controlled invariant domain contained in } R[G, \rho].$$

This domain is the analogous of the maximal (A,B)-invariant subspace of the geometric approach [15], but in the case of an imposed polyhedral domain $R[G, \rho]$ in the state space, instead of an imposed subspace.

4.2 The $l^1$-optimal control problem and the maximal controlled invariant domain

Under the assumption that the set $C_\infty(S(C, \gamma 1))$ is compact, the following result has been established [14]:

**Proposition 3** There exists a controller which achieves a performance of $\gamma$ for system (1), (2) if and only if $C_\infty(S(C, \gamma 1))$ is not empty.

In most practical cases, the number of controlled outputs, $p$ is less than the dimension of the state space, $n$. Then, the polyhedral domain $S(C, \gamma 1)$ has infinite directions, defined by the subspace $\text{Ker } C$. The maximal controlled invariant domain contained in $S(C, \gamma 1)$ may also be unbounded in some directions. Then, its controlled invariance does not guarantee the existence of a stabilizing control achieving the $l^1$ performance bound $\gamma$.

Therefore, two conditions should be imposed to the controller to achieve an $l^1$ performance bound:

1. There exists a controlled invariant domain $\Omega$ containing the zero state and included in $S(C, \gamma 1)$

2. There exists a control sequence $(u_k)$ meeting the first requirement and stabilizing the deterministic system:

$$x_{k+1} = Ax_k + B_2 u_k$$

(24)

Under the assumption that matrix $C$ has rank $n$, the general technique for computing the optimal $l^1$ performance, denoted $\gamma^*$ consists of iteratively computing a sequence of maximal controlled invariant sets (also called the controlled invariant kernels) related to decreasing values of $\gamma$, until obtaining an empty set for the first value smaller than $\gamma^*$. As pointed out in Shamma
[14], this algorithm has several serious drawbacks. In particular, the critical step of the algorithm, which is the construction of the maximal controlled invariant set $C_\infty(S(C, \gamma_1))$ for a given value of $\gamma$, usually requires much computation. As it will now be shown, the results on controlled invariance presented in section 3 are useful for decreasing the computational burden of the algorithm.

### 4.3 Computation of maximal controlled invariant sets

From Proposition 1, the maximal one step admissible domain, $Q(R[G, \rho])$, coincides with the polyhedral domain $R[TGA, T(\rho - \delta)]$. To construct the maximal controlled invariant domain, it suffices then to construct the sequence of sets \{C_i\}, with $C_1 = R[G, \rho]$, and

$$C_{i+1} = Q(C_i) \cap C_i$$

(25)

to obtain:

$$C_\infty(R[G, \rho]) = \lim_{i \to \infty} C_i.$$

It is clear from this construction that $C_\infty(R[G, \rho]) = Q(C_\infty(R[G, \rho])) \subset R[G, \rho]$.

Construction principles have been proposed in [9], [11], [2] for generating this set. Such construction usually requires much computation. In the course of the iterative process, many redundant constraints may be generated. It is therefore particularly desirable to construct an algorithm which only generates non-redundant constraints at each iteration. This property can be obtained using the controlled invariance relations (16), (17). To this end, the algorithm proposed in [7] can be specialized to the computation of $C_\infty(S(C, \gamma_1))$ as follows.

1. Set $i = 1$, $l_0 = 0$, $T^0 = 0$, $t_0 = 0$, $Q^1 = C$, $\phi^1 = \gamma_1$, $g_1 = g$.

2. Compute matrix $T^i \in \mathbb{R}^{t_i \times g_i}$, and decompose $T^i$ as follows:

$$T^i = \begin{bmatrix} T^{i-1} & 0_{t_{i-1} \times t_{i-1}} \\ U^i & \end{bmatrix}; \quad U^i \in \mathbb{R}^{r_i \times g_i}$$

(26)

3. Solve by Linear Programming the following problems, for $j = 1, \ldots, r_i$:

$$\text{minimize } \epsilon^i_j$$

subject to

$$\left\{ \begin{array}{l}
|Y^i_1 Q^i j| \phi^i + |U^j| |Q^i B_1| 1 - \epsilon^i_j |U^j| \phi^i \leq 0 \\
Y^i_1 Q^i j = U^i_1 Q^i A.
\end{array} \right.$$ 

(27)

Set $\epsilon^i = \max_{j=1,\ldots,r_i} \epsilon^i_j$.

If $\epsilon^i \leq 1$, then $C_\infty(S(C, \gamma_1)) = S(Q^i, \phi^i)$, stop.

If $\epsilon^i > 1$, decompose matrices $U^i$ and $Y^i$, using a permutation matrix $P_i$, into the forms

$$P_i U^i = \begin{bmatrix} U^i_1 \\ U^i_2 \end{bmatrix}, \quad P_i Y^i = \begin{bmatrix} Y^i_1 \\ Y^i_2 \end{bmatrix}, \quad \text{with } Y^i_2 \in \mathbb{R}^{r_i \times r_i},$$

such that

$$|Y^i_1| \phi^i \leq |U^i_1| (\phi^i - |Q^i B_1| 1),$$

(29)

$$|Y^i_2| \phi^i > |U^i_2| (\phi^i - |Q^i B_1| 1).$$

(30)
4. Construct the matrices 
\[
Q_{i+1}^i = \begin{bmatrix} Q^i \\ U_i^2 Q^i A \end{bmatrix}.
\]
\[
\phi_{i+1}^i = \begin{bmatrix} U_2^i (\phi^i - |Q^i B_1|) \\ |U_2^i| \end{bmatrix}.
\]
Set \( g_{i+1} = g_i + l_i \).

5. set \( i = i + 1 \) and return to step 2.

The fact that the proposed algorithm converges to \( C_\infty(S(C, \gamma 1)) \) derives from the following result:

**Proposition 4** The polyhedral set \( S(Q^{i+1}, \phi^{i+1}) \) is identical to the set \( Q(S(Q^i, \phi^i)) \cap S(Q^i, \phi^i) \).

**Proof:** From Proposition 1 and (26), the set \( Q(S(Q^i, \phi^i)) \cap S(Q^i, \phi^i) \) is given by the set of constraints:
\[
\begin{cases}
|Q^i x| \leq \phi^i \\
|U_1^i Q^i A x| \leq |U_1^i| (\phi^i - |Q^i B_1|) \\
|U_2^i Q^i A x| \leq |U_2^i| (\phi^i - |Q^i B_1|)
\end{cases}
\]

Note that the rows corresponding to \( T_{i-1} \) in (26) have already been considered in the computation of \( S(Q^i, \phi^i) \).

Inclusion of \( Q(S(Q^i, \phi^i)) \cap S(Q^i, \phi^i) \) in \( S(Q^{i+1}, \phi^{i+1}) \) is obvious. Conversely, any point \( x \in S(Q^{i+1}, \phi^{i+1}) \) satisfies, with nonnegative matrix \( Y_1^i \) satisfying (28), (29):
\[
|U_1^i Q^i A x| = |Y_1^i Q^i x| 
\leq |Y_1^i| \phi^i 
\leq |U_1^i| (\phi^i - |Q^i B_1|)
\]
which implies \( S(Q^{i+1}, \phi^{i+1}) \subset Q(S(Q^i, \phi^i)) \cap S(Q^i, \phi^i) \). \( \square \)

5. Some results when the sets \( C_\infty(S(C, \gamma 1)) \) are not bounded

In this section, the rank of matrix \( C \) may be less than or equal to \( n \). The analysis will first study the case for which \( S(C, \gamma 1) \) is controlled invariant.

### 5.1 Conditions under which \( S(C, \gamma 1) \) is controlled invariant

By application of Corollary 1 with \( D = S(I_p, 1) \), a necessary and sufficient condition for controlled invariance of \( S(C, \gamma 1) \) with respect to system (1)-(3), is:
\[
\gamma \geq \delta_i \text{ with } \delta_i = \max_{\|w\|_\infty \leq 1} C_i B_1 w = |C_i B_1| \text{ for } i = 1, \ldots, p
\]
and the existence of a matrix \( Y \) such that:
\[
Y C = T C A, \quad Y |\gamma 1| \leq |T| (|\gamma 1| - \delta), \text{ with } \delta = |C B_1| 1.
\]

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If \( p = \text{rank}(C) < n \), condition (34) can be satisfied if and only if \( \text{Ker } C \) is an \((A,B)\)-invariant subspace of the deterministic system (24) (see [8]). The properties of \((A,B)\)-invariant subspaces have been extensively studied [15]. In particular, a linear control, \( u_k = F x_k \) may achieve invariance of \( \text{Ker } C \) with respect to system (24). A basic requirement for \((A+BF)\)-invariance of \( \text{Ker } C \) is to use the invariant zeros (if they exist) of system \((A, B_2, C)\) as closed-loop eigenvalues. Therefore, because of the asymptotic stability requirement, such a placement is possible only if the zeros of system \((A, B_2, C)\) are asymptotically stable, that is if this system is minimum phase.

It has also be shown [10] that \((A,B)\)-invariance of \( \text{Ker } C \) is a necessary condition for controlled invariance of the polyhedral domain \( S(C,1) \), with respect to the deterministic system (24). Let us now assume that the polyhedral domain \( S(C,1) \) is controlled invariant with stability, with respect to (24). This property can be characterized by the existence of a pair of matrices \((F,H)\) and a nonnegative vector \( \omega \leq 1 \) such that [10] :

\[
(A + B_2 F) \text{ is asymptotically stable} \quad (36)
\]

\[
HC = C(A + B_2 F) \quad (37)
\]

\[
|H| \leq \omega \quad (38)
\]

Then, the following property can be used.

**Proposition 5** If \( S(C,1) \) is controlled invariant with respect to system (24), then \( S(C, \gamma 1) \) is controlled invariant with respect to system (1) if the following condition is satisfied, with \( \delta = |CB_1|1 \) :

\[
\omega \leq 1 - \frac{1}{\gamma} \delta \quad (39)
\]

Furthermore, under condition (39) and if system \((A,B_2,C)\) is minimum phase, the linear control law \( u_k = F x_k \), with \((F,H)\) solution of (36), (37), (38), can be applied to (1) to achieve the \( l^1 \) performance bound \( \gamma \).

**Proof:** Left-multiplication of both terms in (37) by matrix \( T \) defined before Corollary 1, yields:

\[
T HC = TCA
\]

and left-multiplication of both terms in (38) by \(|T|\) yields :

\[
|T||H| \leq |T|\omega
\]

and thus, for \( Y = TH \) and using (39),

\[
YC = TCA \quad (40)
\]

\[
|Y| \leq |T||H| \gamma 1 \leq |T|\gamma \omega \leq |T|(\gamma 1 - \delta) \quad (41)
\]

Then, \( S(C,1) \) is invariant with respect to (1) under the control \( u_k = F x_k \).

\[\Box\]
5.2 The $l^1$ optimal control problem in a special case

Proposition 5 will now be used to show that, for a particular class of systems, the solution of the $l^1$ optimal control problem is straightforward. This class contains as a subclass the monovariable minimum phase case for which a simple and explicit solution to the the $l^1$ optimal control problem has been proposed in [6]. The considered case is relatively frequent in practice. It is characterized by the following hypothesis:

Hypothesis $(H_1)$:

$\text{rank}(CB_2) = \text{rank}(C) = p \leq m$, the pair $(A, B_2)$ is controllable and system $(A, B_2, C)$ is minimum phase.

The following Proposition can be established

Proposition 6 Under hypothesis $H_1$, the optimal $l^1$ performance value is:

$$\gamma^* = \|CB_1\|_\infty. \quad (42)$$

It can be achieved by a constant state feedback law $u_k = Fx_k$ under which, for the associated deterministic system (24), the subspace $\text{Ker } C$ is invariant with stability and the deadbeat response is achieved in the quotient subspace $(\mathbb{R}^n/\text{Ker } C)$.

Proof: It has been shown ([10], Proposition IV.2) that under hypothesis $H_1$, the subspace $\text{Ker } C$ is $(A, B)$-invariant with stability with respect to system (24). Let $F_0$ be a friend of $\text{Ker } C$ (see [15]), with the restriction of $(A + B_2F_0)$ to $\text{Ker } C$ asymptotically stable. Any feedback gain matrix

$$F = F_0 + F_1C \quad (43)$$

is also a friend of $\text{Ker } C$, and the restriction $(A + B_2F)|\text{Ker } C$ is identical to $(A + B_2F_0)|\text{Ker } C$. The restriction of $(A + B_2F_0)$ to $(\mathbb{R}^n/\text{Ker } C)$ is denoted $\tilde{A}_0$ and defined by the canonical projection equation:

$$\tilde{A}_0C = C(A + B_2F_0) \quad (44)$$

Define $\tilde{B} = CB_2$. Under the assumption that $(A, B_2)$ is controllable, then $(\tilde{A}_0, \tilde{B})$ is also controllable in $(\mathbb{R}^n/\text{Ker } C)$ (see [15]). Moreover, with $\text{rank}(\tilde{B}) = \text{rank}(C) = p \leq m$, the closed-loop projection in the quotient subspace $(\mathbb{R}^n/\text{Ker } C)$ can be selected to achieve the deadbeat response:

$$(\tilde{A}_0 + \tilde{B}F_1) = 0$$

To do so, it suffices to select $F_1$ such that:

$$\tilde{B}F_1 = -\tilde{A}_0 \quad (45)$$

And in particular,

$$F_1 = -\tilde{B}^T(\tilde{B}\tilde{B}^T)^{-1}\tilde{A}_0 \quad (46)$$

Then, under the control law $u_k = Fx_k$,

$$C(A + B_2F) = (\tilde{A}_0 + \tilde{B}F_1)C = 0 \quad (47)$$

A pair of matrices $(H, F)$ and a vector $\omega$ satisfying conditions (36), (37), (38) are thus obtained for $F$ given by (43), $H = 0$ and $\omega = 0$. 

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The effects of the disturbance term are now considered. Proposition 4 applied for $\omega = 0$ shows that the $l^1$ performance value obtained with the constant feedback gain matrix $F$ (43) is

$$\gamma = \| CB_1 \|_{\infty}.$$ 

As this value measures the maximal direct effect of the disturbance inputs on the current system state, it is not possible to achieve a smaller value of $\gamma$ under the assumption that the disturbance inputs cannot be measured. Therefore, the obtained $l^1$ performance value is optimal.

\[\square\]

### 5.3 Numerical Example

A very simple numerical example is given below, as an example of a minimum phase system for which the $l^1$ Optimal Control Problem admits a very simple solution. The considered system is defined by

$$A = \begin{bmatrix} 0.48 & 0.88 & -0.77 \\ -0.61 & 0.65 & -0.96 \\ -0.27 & 0.49 & 0.99 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.53 \\ -0.35 \\ 0.44 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -0.84 & 0.42 \\ 0.75 & -0.26 \\ -0.04 & 0.24 \end{bmatrix}, \quad C = \begin{bmatrix} -0.65 & 0.77 & -0.99 \\ 0.49 & -0.96 & -0.84 \end{bmatrix}.$$ 

System $(A, B_2, C)$ has one stable invariant zero : $z \simeq 0.66$. In this example, matrix $CB_2$ is square and regular. The invariant zero of $(A, B_2, C)$ is automatically placed as a closed-loop eigenvalue under $u_k = Fx_k$ with matrix $F$ uniquely given by:

$$F = -(CB_2)^{-1}CA = \begin{bmatrix} 1.2655 & -1.1764 & -1.0583 \\ 1.3470 & -2.7081 & -3.4464 \end{bmatrix},$$

This state feedback control achieves the optimal $l^1$ performance value:

$$\gamma^* = \| CB_1 \|_{\infty} = 1.0496.$$ 

### 6 Conclusion

This paper has shown that controlled invariance properties are very useful for solving some $l^1$ optimal control problems. Two major contributions have been proposed:

- improved computation of the maximal controlled invariant domain achieving a given $l^1$ performance value,
- construction of a simple solution to the $l^1$ optimal control problem for a class of minimum phase systems.

An interesting problem still under investigation is to construct the maximal controlled invariant domain with stability in the case of non minimum phase systems.

### References


