A LINEAR PROGRAMMING APPROACH FOR REGIONAL POLE PLACEMENT UNDER POINTWISE CONSTRAINTS

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Abstract

This paper considers the problem of control of discrete-time linear systems under several complicating features frequently encountered in practice: uncertain parameters in the model, additive noise, linear symmetrical state and input constraints. The main control design objective, beyond local stabilization, is to locate the closed-loop poles in a “good” region of the unit disc of the complex plane. The originality of the proposed approach is to use Linear Programming as the basic design tool. It is shown that the matrix conditions derived from positive invariance relations and from design constraints can be formulated as linear matrix equalities and scalar inequalities. The control design problem can then be solved by Linear Programming, with an objective function describing the search for a trade-off between several design objectives.

Keywords

State Feedback, Disturbance Rejection, Linear systems, Invariance, Uncertain systems.

1 Introduction

Many practical control problems are characterized by several complicating features in the system model, such as persistent disturbances, parameters uncertainty, pointwise constraints on input and output variables. Moreover, the control objectives often go beyond stabilization or disturbance attenuation, seeking for a satisfactory trade-off between performance and robustness.

When the system model admits a linear description in a limited domain of the state space, most of these features and requirements can be translated into linear matrix conditions. Using the approach by positive invariance of polyhedral domains, only linear matrix equalities and scalar inequalities are involved. The use of Linear Programming algorithm is then possible, with the advantage of a great computational efficiency.

This paper describes this approach in the case of a discrete-time linear model with additive disturbances and convex parameters uncertainties, subject to symmetrical linear constraints on its input vector. The addressed control problem is of the $l^1$-type with additional constraints and pole location requirements for the closed-loop state matrix. Classically [13], the effects of the additive disturbance on the system output are measured in terms of the $l^\infty$-induced norm. Here, they will be represented

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by pointwise symmetrical constraints on the output vector. Additionally, a major concern for the considered control problem is to locate the closed-loop poles in a particular region of the unit disc of the complex plane. Linear formulations of this problem are provided in the cases of pole regions made of the intersection of the unit disc with a vertical strip and of centered rings in the unit disc.

Notations. The following notations are used throughout the paper. Matrix $I_n$ is the identity matrix in $\mathbb{R}^{n \times n}$; vector $1_n$ is defined as $1_n = [1 \ldots 1]^T \in \mathbb{R}^n$. For any matrix $M \in \mathbb{R}^{n \times n}$, $|M|$ denotes the matrix of the absolute values of its components: $|M|_{ij} = |M_{ij}|$. $\hat{M}$ is defined by:

$$
\hat{M}_{ii} = M_{ii}, \quad \hat{M}_{ij} = |M_{ij}| \text{ for } j \neq i.
$$

2 Problem statement

2.1 Linear control under output and input constraints

Consider the linear discrete-time system represented by the following time-invariant equations:

$$
\begin{align*}
x_{k+1} &= Ax_k + Bu_k + Dd_k, \quad \forall k \in \mathbb{N} \\
y_k &= Gx_k
\end{align*}
$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, with $\text{rank}(B) = m$, $D \in \mathbb{R}^{n \times l}$, $G \in \mathbb{R}^{g \times n}$, $\text{rank}(G) = n$. Vectors $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $d_k \in \mathbb{R}^l$ and $y_k \in \mathbb{R}^g$ are, respectively, the current state, control, disturbance and output vectors, at time $k$.

As in most practical cases, all the variables of this model are subject to constraints. In the proposed model, all the constraints are assumed to be linear and symmetrical:

- the disturbance vector, $d_k$, is assumed to belong to a closed convex set:

$$
d_k \in S(P, \xi) \subset \mathbb{R}^l ; \forall k \in \mathbb{N} \\
S(P, \xi) = \{d \in \mathbb{R}^l ; \xi \leq Pd \leq \xi\}
$$

with: $P \in \mathbb{R}^{p \times l}$, $\xi \in \mathbb{R}^p$, $\xi > 0$;

- the control inputs are assumed to be symmetrically bounded:

$$
u_k \in \mathcal{U} \subset \mathbb{R}^m ; \forall k \in \mathbb{N} \\
\mathcal{U} = \{u \in \mathbb{R}^m ; -\rho \leq u_k \leq \rho\}
$$

with: $\rho \in \mathbb{R}^m$ and $\rho > 0$;

- the disturbance rejection requirement imposes that the states of the system should remain in a compact symmetrical polyhedron, denoted $S(G, \gamma)$:

$$
x_k \in S(G, \gamma) \subset \mathbb{R}^n ; \forall k \in \mathbb{N} \\
S(G, \gamma) = \{x \in \mathbb{R}^n ; -\gamma \leq Gx_k \leq \gamma\}
$$

with $\gamma \in \mathbb{R}^g$, $\gamma > 0$. 


A class of candidate solutions of this problem is constructed with a linear state feedback control law
\[ u_k = F x_k \]  
(6)
such that the disturbed linear closed-loop model
\[ x_{k+1} = (A + BF)x_k + D e_k \]  
(7)
admits a set \( \Omega \) that is \( S(P, \xi) \)-invariant in the state space and satisfies [14]) :
\[
\left\{ \begin{array}{l}
\Omega \subset S(G, \gamma) \\
\Omega \subset S(F, \rho) = \{ x \in \mathbb{R}^n ; \ -\rho \leq F x_k \leq \rho \} 
\end{array} \right. 
\]

**Definition 1** [3] A set \( \Omega \subset \mathbb{R}^n \) is \( S(P, \xi) \)-invariant, if for any initial condition belonging to \( \Omega \) the corresponding trajectory remains in \( \Omega \) for any sequence of disturbances \( d \in S(P, \xi) \). \( \square \)

The particular choice of \( S(G, \gamma) \) as the candidate set \( \Omega \) corresponds to the following conditions :
\[
\exists H \in \mathbb{R}^{g \times g}, \ \exists K \in \mathbb{R}^{g \times p}, \ \exists M \in \mathbb{R}^{m \times g} \text{ such that :} 
\]
\[
H G = G A + G B M G \]  
(8)
\[ K P = G D \]  
(9)
\[ |M| \gamma \leq \rho \]  
(10)
\[ |H| \gamma + |K| \xi \leq \gamma \]  
(11)

If these conditions are satisfied, the closed-loop system trajectory satisfies constraints (4), (5) from any initial state \( x_0 \in S(G, \gamma) \) and for any sequence of disturbances \( d_k \in S(P, \xi) \), under the control law (6) with \( F = MG \).

### 2.2 Constraints on the closed-loop pole location

Clearly, \( S(P, \xi) \)-invariance of a set \( \Omega \) with respect to system (7) is a sufficient condition for positive invariance of \( \Omega \) with respect to the associated deterministic system (12) defined by :
\[ x_{k+1} = (A + BF)x_k. \]  
(12)

Then, classically [1], [7], positive invariance of a compact domain in the state space having the zero state in its interior is a sufficient condition for local stability of the constrained deterministic system around the zero state. This property implies, in particular that, under conditions (8), (9), (10), (11), the closed-loop state matrix \( (A + BF) \) has its poles located in the unit circle of the complex plane. This elementary stability condition may not be sufficient for the designer. In many practical applications, it is desirable to locate the closed-loop poles in a particular region, \( \Sigma \), of the unit circle, and thus to impose the additional spectral constraints :
\[ \sigma(A + BF) \subset \Sigma \]  
(13)

Two types of spectral regions \( \Sigma \) will be considered in the sequel. The associated regional pole location conditions will be characterized and incorporated in the control design problem, either as hard constraints or as design objectives.
2.3 Model uncertainties

If some of the model parameters in (1) are uncertain, the system dynamics can be represented by a family of models (1):

\[ A \in A ; \quad B \in B \]  

A general representation of uncertainties (see e.g. [6], [10]) is obtained when assuming that the sets \( A \) and \( B \) are convex and compact polyhedral domains defined by their vertices \((A_i \in \mathbb{R}^{n \times n}, B_j \in \mathbb{R}^{n \times m})\), through the following relations:

\[ A = \{ A : A = \sum_{i=1}^{N_A} \zeta_i A_i, \zeta_i \geq 0, \sum_{i=1}^{N_A} \zeta_i = 1 \} \]  \hspace{1cm} (15)

\[ B = \{ B : B = \sum_{j=1}^{N_B} \kappa_j B_j, \kappa_j \geq 0, \sum_{j=1}^{N_B} \kappa_j = 1 \} \]  \hspace{1cm} (16)

If \( na \) and \( nb \) denote the numbers of uncertain entries in \( A \) and \( B \), the number of vertices defining the convex polyhedral domain of uncertainties for the pair \((A, B)\) has the order of magnitude \( 2^{na+nb} \). Therefore, it is mainly for small values of \( na + nb \) that such a representation of uncertainties may be considered.

It has been shown by B.E.A.Milani, A.N.Carvalho [10] (see also [11]) that due to the convexity and compactness of the domain of uncertainties, a necessary and sufficient condition for relations (8), (9), (11) to be satisfied by any possible pair of matrices \((A, B)\) in the domain of uncertainties is that for any pair of vertices \((A_i, B_j)\):

\[ \exists H_{ij} \in \mathbb{R}^{q \times g}, \exists K \in \mathbb{R}^{g \times p} \text{ such that:} \]

\[ H_{ij} G = GA_i + GB_j F \]  \hspace{1cm} (17)

\[ KP = GD \]  \hspace{1cm} (18)

\[ |H_{ij}| \gamma + |K| \xi \leq \gamma \]  \hspace{1cm} (19)

It is not difficult to show a similar result for the pole placement relation (13) : \( \sigma(A + BF) \subset \Sigma \) for any possible pair of matrices \((A, B)\) if and only if for any pair of vertices \((A_i, B_j)\), the following relation is verified:

\[ \sigma(A_i + B_j F) \subset \Sigma. \]  \hspace{1cm} (20)

3 Some linear representations of regional pole placement

The basic technique which will be used for obtaining sufficient pole placement conditions is derived from the two following stability theorems due to A.P. Molchanov, V.S. Pyatniskyi [12]. In the sequel, \( \lambda_i(H) \) denotes the \( i \)th eigenvalue of a matrix \( H \) and \( |\lambda_i(H)| \) the corresponding modulus

**Theorem 1** [12] : The linear time-invariant discrete-time system \( x_{k+1} = Hx_k \) is asymptotically stable, that is, \( |\lambda_i(H)| < 1 \) for \( i = 1, \ldots, n \), if and only if there exist a matrix \( Q \in \mathbb{R}^{n \times n} \) with rank\((Q) = n \) and a matrix \( \Gamma \in \mathbb{R}^{q \times q} \), such that:

\[ \Gamma Q = QH \]  \hspace{1cm} (21)

\[ \|\Gamma\|_\infty < 1 \]  \hspace{1cm} (22)
Theorem 2 \cite{12} : The linear time-invariant continuous-time system \( \dot{x}(t) = Hx(t) \) is asymptotically stable, that is, \( \Re(\lambda_i(H)) < 0 \) for \( i = 1, \ldots, n \), if and only if there exist a matrix \( Q \in \mathbb{R}^{q \times n} \) with \( \text{rank}(Q) = n \) and a matrix \( \Gamma \in \mathbb{R}^{q \times q} \), such that:

\[
\Gamma Q = QH
\]
\[
\mu_\infty(\Gamma) < 0 \tag{24}
\]

The following remarks can be made about the above theorems:

- Equations (21) and (23) can be seen as generalized similarity relations. In particular, since \( \text{rank}(Q) = n \), we have:

\[
\sigma(H) \subseteq \sigma(\Gamma) \tag{25}
\]

Furthermore, from basic properties of matrix norms and matrix measures, inequalities (22) and (24) guarantee the asymptotic stability of the eigenvalues of \( \Gamma \) (respectively in the discrete-time and the continuous-time cases) and, in consequence of (25), the asymptotic stability of the eigenvalues of \( H \).

- For computational purposes, equations (22) and (24) can be respectively replaced by:

\[
|\Gamma|_q \leq 1 
\]
\[
\hat{\Gamma} 1_q \leq 0 
\]

- Candidate Lyapunov functions are described by:

\[
V(x) = \max_i |(Qx)_i|
\]

The corresponding domains of stability and invariance are convex and polyhedral.

### 3.1 Pole location in a vertical strip

Let \( \mathcal{F}(\alpha, \beta) \) represent a vertical strip in the complex plane, where \( \alpha \) and \( \beta \) are real numbers satisfying \( \alpha > \beta \):

\[
\lambda_i(H) \in \mathcal{F}(\alpha, \beta) \iff \beta < \Re[\lambda_i(H)] < \alpha \tag{26}
\]

The objective is to obtain algebraic conditions which guarantee that the eigenvalues of a matrix \( H \) belong to \( \mathcal{F}(\alpha, \beta) \). The following proposition gives necessary and sufficient conditions for (26) to be verified.

**Proposition 1** The eigenvalues of a matrix \( H \in \mathbb{R}^{n \times n} \) verify \( \lambda_i(H) \in \mathcal{F}(\alpha, \beta) \), if and only if there exist matrices \( Q_1 \in \mathbb{R}^{m_1 \times n} \) and \( Q_2 \in \mathbb{R}^{m_2 \times n} \) with \( \text{rank}(Q_1) = \text{rank}(Q_2) = n \) and matrices \( \Gamma_1 \in \mathbb{R}^{m_1 \times m_1} \) \( \Gamma_2 \in \mathbb{R}^{m_2 \times m_2} \) such that:

\[
\Gamma_1 Q_1 = Q_1 H 
\]
\[
\mu_\infty(\Gamma_1) < \alpha \tag{27}
\]
\[
\Gamma_2 Q_2 = Q_2 H 
\]
\[
\mu_\infty(-\Gamma_2) < -\beta \tag{29}
\]

**Proof :** It will first be shown that the existence of matrices \( Q_1 \) and \( \Gamma_1 \) satisfying (27) and (28) is a necessary and sufficient condition for verifying \( \Re[\lambda_i(H)] < \alpha \). To this end, consider the matrix \( H_\alpha = H - \alpha I \), whose eigenvalues verify \( \lambda_i(H_\alpha) = \lambda_i(H) - \alpha \). Then, \( \Re[\lambda_i(H)] < \alpha \) if and only if
\[ \Re[\lambda_i(H)] < 0. \] From Theorem 2, this is verified if and only if there exist \( Q_1 \in \Re^{m_1 \times n} \) of rank \( n \), and \( \Gamma_1 \in \Re^{m_1 \times m_1} \) such that

\[
\Gamma_1 Q_1 = Q_1 H \alpha \quad (31)
\]

\[ \mu_\infty \Gamma_1 < 1 \quad (32) \]

Thus, equations (27) and (28) are obtained from (31) and (32) by defining \( \Gamma_1 = \Gamma_1 + \alpha I_n \) and by using the fact that \( \mu_\infty (\bar{\Gamma}_1) = \mu_\infty (\Gamma_1) + \alpha \).

To complete the proof, it must be shown that the verification of (29) and (30) is a necessary and sufficient condition for \( \Re[\lambda_i(H)] > \beta \) to be satisfied. This can be done by remarking that \( \Re[\lambda_i(H)] > \beta \) is equivalent to \( \Re[\lambda_i(-H)] < -\beta \), which is verified if and only if there exist \( m_2 > n \), \( Q_2 \in \Re^{m_2 \times n} \) of full rank, and \( \bar{\Gamma}_2 \in \Re^{m_2 \times m_2} \) such that

\[
\bar{\Gamma}_2 Q_2 = Q_2 (-H) \quad (33)
\]

\[ \mu_\infty (\bar{\Gamma}_2) < -\beta \quad (34) \]

Thus, (33) and (34) can be equivalently replaced by (29) and (30) when setting \( \bar{\Gamma}_2 = -\bar{\Gamma}_2 \). □

Notice that even for a given \( H \), relations (27) and (29) are non-linear and cannot be used as constraints in linear programming. The result that will be used in the sequel for characterizing the location of the eigenvalues of \( H \) in \( \mathcal{F}(\alpha, \beta) \) is the following one:

**Property 1** If \( \mu_\infty (H) < \alpha \) and \( \mu_\infty (-H) < \beta \) then \( \lambda_i(H) \in \mathcal{F}(\alpha, \beta) \). □

### 3.2 Pole location in a centered ring

Let \( \mathcal{A}(\delta, \tau) \) define a centered ring in the complex plane, where \( \delta \) and \( \tau \) are real numbers satisfying \( \delta > \tau > 0 \):

\[
\lambda_i(H) \in \mathcal{A}(\delta, \tau) \iff \left\{ \begin{array}{l}
\rho(H) < \delta \\
\varrho(H) > \tau
\end{array} \right. \quad (35)
\]

where: \( \rho(H) = \max_i |\lambda_i(H)| \) e \( \varrho(H) = \min_i |\lambda_i(H)| \).

Similarly to the previous case, the objective is to derive algebraic conditions which guarantee that the eigenvalues of a matrix \( H \) belong to \( \mathcal{A}(\delta, \tau) \). Since \( \tau > 0 \) guarantees the existence of \( H^{-1} \), we have: \( \lambda_i(H^{-1}) = \frac{1}{\lambda_i(H)} \), which implies \( \varrho(H) = \frac{1}{\rho(H^{-1})} \). Thus, (35) can be replaced by:

\[
\lambda_i(H) \in \mathcal{A}(\delta, \tau) \iff \left\{ \begin{array}{l}
\rho(H) < \delta \\
\rho(H^{-1}) < \frac{1}{\tau}
\end{array} \right. \quad (36)
\]

The following result gives a necessary and sufficient condition for (36) to be verified:

**Proposition 2** The eigenvalues of a matrix \( H \in \Re^{n \times n} \) verify \( \lambda_i(H) \in \mathcal{A}(\delta, \tau) \), if and only if there exist matrices \( Q_1 \in \Re^{m_1 \times n} \) and \( Q_2 \in \Re^{m_2 \times n} \), with \( \text{rank}(Q_1) = \text{rank}(Q_2) = n \) and matrices \( \Gamma_1 \in \Re^{m_1 \times m_1} \) and \( \bar{\Gamma}_2 \in \Re^{m_2 \times m_2} \) such that:

\[
\Gamma_1 Q_1 = Q_1 H \quad (37)
\]

\[ \|\Gamma_1\|_\infty < \delta \quad (38) \]

\[
\bar{\Gamma}_2 Q_2 = Q_2 H^{-1} \quad (39)
\]

\[ \|\bar{\Gamma}_2\|_\infty < \frac{1}{\tau} \quad (40) \]
Proof: Consider the matrix \( H_\delta = \frac{1}{\delta} H \), with \( \delta > 0 \), whose eigenvalues satisfy \( \lambda_i(H_\delta) = \frac{1}{\delta} \lambda_i(H) \). Then, \( \rho(H_\delta) < 1 \) if and only if \( \rho(H) < \delta \), which is verified if and only if there exist \( Q_1 \in \mathbb{R}^{m \times n} \) of rank \( n \), and \( \Gamma_1 \in \mathbb{R}^{m \times m_1} \) such that:

\[
\Gamma_1 Q_1 = Q_1 H_\delta \quad (41)
\]

\[
\|\Gamma_1\|_\infty < 1 \quad (42)
\]

Relations (37) and (38) are obtained from (41) and (42) by defining \( \Gamma_1 = \frac{1}{\delta} \Gamma_1 \) and by using the fact that \( \|\Gamma_1\|_\infty = \frac{1}{\delta} \|\Gamma_1\| \), because \( \delta \) is positive. To complete the proof, it suffices to apply (41) and (42) to obtain \( \rho(H^{-1}) < \frac{1}{\delta} \).

As in the case of vertical strips, it is possible to derive sufficient conditions for characterizing the location of eigenvalues in \( \mathcal{A}(\delta, \tau) \). In addition, the following result (see [8], page 167) can be used:

Lemma 1: If \( Z \in \mathbb{R}^{w \times w} \) satisfy:

\[
0 < |z_{ii}| - \sum_{j \neq 1}^w |Z_{ij}| = \alpha_i \quad \forall i = 1, ..., w
\]

(that is, \( Z \) is strictly diagonally dominant by rows), then

\[
\rho(Z^{-1}) \leq \|Z^{-1}\|_\infty \leq \frac{1}{\min_i \alpha_i}
\]

Thus, the following characterization is obtained for the location of the eigenvalues of \( H \) in \( \mathcal{A}(\delta, \tau) \):

Property 2: If \( \|H\|_\infty < \delta \) and \( \min_i |H_{ii}| - \sum_{i \neq j} |H_{ij}| > \tau \) then \( \lambda_i(H) \in \mathcal{A}(\delta, \tau) \).

4 A Linear Programming Formulation of the Control Problem

4.1 An LP formulation to satisfy state and input constraints

The basic constrained control problem which has been stated in section 2.1 may be solved by finding matrices \( (H, K, M) \) which satisfy relations (8), (9), (10), (11). The two first relations are linear matrix equalities. The third one and the fourth one involve the absolute values of matrices \( H \), \( K \) and \( M \). A classical way to linearly reformulate relation (11) is through the decomposition of these matrices into their positive and negative elements (see e.g. [7]). Any matrix \( L \) and, in consequence, the matrix \( |L| \), can be decomposed into:

\[
L = L^+ - L^- \quad \text{and} \quad |L| = L^+ + L^-
\]

with: \( L^+_{ij} = \max(L_{ij}, 0) \) and \( L^-_{ij} = \max(-L_{ij}, 0) \).

Using matrices \( H^+, H^-, K^+, K^-, M^+, M^- \), relations (10), (11) are then formulated as linear relations, and the satisfaction of the second relation in (43) is forced by the choice of the objective function.

\[
L = L^+ - L^- \quad \text{and} \quad |L| = L^+ + L^-
\]

with: \( L^+_{ij} = \max(L_{ij}, 0) \) and \( L^-_{ij} = \max(-L_{ij}, 0) \).

Using matrices \( H^+, H^-, K^+, K^-, M^+, M^- \), relations (10), (11) are then formulated as linear relations, and the satisfaction of the second relation in (43) is forced by the choice of the objective function.
The LP formulation of the constrained disturbance rejection problem takes the following form, with the components of matrices $H^+, H^-, K^+, K^-, M^+, M^-$ as unknown variables:

Minimize \[ p_1 \epsilon + p_2 \zeta \]
under \[
(H^+ - H^-)G - GB(M^+ - M^-)G = GA \\
(K^+ - K^-)P = GD \\
(M^+ + M^-) \gamma - \zeta \rho \leq 0 \\
(H^+ + H^-) \gamma + (K^+ - K^-) \xi - \epsilon \gamma \leq 0 \\
0 \leq \epsilon \leq 1 ; 0 \leq \zeta \leq 1 \\
H^+, H^-, K^+, K^-, M^+, M^- \geq 0
\]

(44)

If this problem has no solution, then the choice $\Omega = S(G, \gamma)$ is not admissible. This does not always mean that the performance bound vector $\gamma$ is impossible to achieve. To be sure that this bound is not achievable, one has to show that the maximal controlled invariant set included in $S(G, \gamma)$ is empty (see [13]). The study of the maximal controlled invariant set included in a given compact polyhedron has been made by several authors [9], [2], and numerical techniques have been proposed for computing this domain [4], [5].

If the model is uncertain, as described by (14), (15) and (16), unknown matrices $H^+_{ij}, H^-_{ij}$, corresponding to each pair of vertices $(A_i, B_j)$, must be computed to verify equations (17) to (19).

4.2 An LP formulation integrating spectral specifications

In order to integrate spectral specifications (pole location in vertical strips, $\mathcal{F}(\alpha, \beta)$, or in centered rings, $\mathcal{A}(\delta, \tau)$) in the design procedure, we propose to add the conditions established in properties 2 and 1 to the LP design procedure.

It is worth noticing that the design methodology is based on the search of a constant state feedback control law which should leave a given polyhedral set invariant with respect to the disturbed system (7). Furthermore, the conditions that will be used for guaranteeing the regional pole location are only sufficient conditions. Thus, in order to provide some additional degrees of freedom in the design technique, we shall consider that:

1) the limits that describe the regions of pole location ($\alpha$ and $\beta$ for vertical strips, $\mathcal{F}(\alpha, \beta)$, and $\delta$ and $\tau$ for centered rings, $\mathcal{A}(\delta, \tau)$) can be used as variables in the LP formulations
2) the linear objective functions constructed from these variables describe different possibilities of locating the poles within regions with the desired shape.

From the above discussion, we propose the following two LP formulations for regional pole location
- Location in vertical strips $\mathcal{F}(\alpha, \beta)$:

\[
\begin{align*}
J &= p_1 \alpha + p_2 \beta \\
\text{subject to } & (H^+ - H^-)G - GB(M^+ - M^-)G = GA \\
& (K^+ - K^-)P = GD \\
& (M^+ + M^-)\gamma \leq \rho \\
& (H^+ + H^-)\gamma + (K^+ - K^-)\xi \leq \gamma \\
& H^i - H^-_i + \sum_{i \neq j} (H^i_{ij} + \sum_{i \neq j} H^-_{ij}) < \alpha \\
& -H^i + H^-_i + \sum_{i \neq j} (H^i_{ij} + H^-_{ij}) < -\beta \\
& \alpha > \beta \\
& H^+, H^-, K^+, K^-, M^+, M^- \geq 0
\end{align*}
\]

(45)

- Location in centered rings $\mathcal{A}(\delta, \tau)$:

\[
\begin{align*}
J &= p_1 \delta + p_2 \tau \\
\text{subject to } & (H^+ - H^-)G - GB(M^+ - M^-)G = GA \\
& (K^+ - K^-)P = GD \\
& (M^+ + M^-)\gamma \leq \rho \\
& (H^+ + H^-)\gamma + (K^+ - K^-)\xi \leq \gamma \\
& (H^+ + H^-)\gamma \leq \delta \gamma \\
& \left[ -(H^i + H^-_i) + \sum_{i \neq j} (H^i_{ij} + H^-_{ij}) \right] 1_n < -\tau 1_n \\
& \delta > \tau \\
& H^+, H^-, K^+, K^-, M^+, M^- \geq 0
\end{align*}
\]

(46)

In both LP formulations given above, the weights $p_1$ and $p_2$ should be used by the designer in order to find a solution that appropriately locates the closed-loop poles in the complex plane..

4.3 A numerical Example

Consider the following matrices describing the model of a DC motor [3] [11]:

\[
A_l = \begin{bmatrix}
-0.0700 & -1.0320 \\
0.0480 & -0.5000
\end{bmatrix} \quad A_u = \begin{bmatrix}
-0.0700 & -0.6880 \\
0.0720 & -0.0085
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1.0000 \\
0.0000
\end{bmatrix}
\]

The elements of matrices $A_l$ and $A_u$ describe, respectively, the lower and upper bounds of each element of the system matrix $A$. Matrix $B$ is supposed perfectly known. For the sake of simplicity, it is assumed that the control input variables are not constrained. Only the disturbance rejection specification is imposed, under the form of a domain of state constraints $S(G, \gamma)$ given by:

\[
G = \begin{bmatrix}
1.0000 & 0.0000 \\
0.0000 & 1.0000
\end{bmatrix} \quad \gamma = \begin{bmatrix}
1.0000 \\
1.0000
\end{bmatrix}
\]
The spectral control objective is to locate the poles of the uncertain closed-loop in a region of type $\mathcal{F}(\alpha, \beta)$, and a constant state feedback solution is investigated.

Using first the linear program (45) with the weights $p_1 = 1$ and $p_2 = -1$, both $\alpha$ and $\beta$ tend to be minimized. The following values of $F$, $\alpha$, $\beta$ have been computed:

$$F = \begin{bmatrix} -0.0385 & 0.8600 \end{bmatrix} ; \quad \alpha = 0.0635 ; \quad \beta = -0.5720$$

The eigenvalues corresponding to each corner $A_{ol} = A_i + BF$, for $i = 1, \ldots, 8$ are the following ones:

$$\lambda_i(A_{o1}) = \begin{bmatrix} -0.1309 \\ -0.4776 \end{bmatrix}, \quad \lambda_i(A_{o2}) = \begin{bmatrix} -0.1884 \\ -0.0051 \end{bmatrix}$$

$$\lambda_i(A_{o3}) = \begin{bmatrix} -0.1432 \\ -0.4653 \end{bmatrix}, \quad \lambda_i(A_{o4}) = \begin{bmatrix} -0.0884 \\ -0.5201 \end{bmatrix}$$

$$\lambda_i(A_{o5}) = \begin{bmatrix} -0.1805 \\ 0.0635 \end{bmatrix}, \quad \lambda_i(A_{o6}) = \begin{bmatrix} -0.0791 \\ -0.5294 \end{bmatrix}$$

$$\lambda_i(A_{o7}) = \begin{bmatrix} -0.0585 \pm 0.0759 \end{bmatrix}$$

$$\lambda_i(A_{o8}) = \begin{bmatrix} -0.0585 \pm 0.0994 \end{bmatrix}$$

Now, using the weights $p_1 = -1$ and $p_2 = -1$, $\alpha$ tends to be maximized while $\beta$ tends to be minimized. The following values of $F$, $\alpha$, $\beta$ have been obtained for this case:

$$F = \begin{bmatrix} 0.0457 & 0.7888 \end{bmatrix} ; \quad \alpha = 1.0000 ; \quad \beta = -0.5720$$

The eigenvalues corresponding to each corner $A_{ol} = A_i + BF$, for $i = 1, \ldots, 8$ are the following ones:

$$\lambda_i(A_{o1}) = \begin{bmatrix} -0.0503 \\ -0.4740 \end{bmatrix}, \quad \lambda_i(A_{o2}) = \begin{bmatrix} 0.0212 \\ -0.1305 \end{bmatrix}$$

$$\lambda_i(A_{o3}) = \begin{bmatrix} -0.0645 \\ -0.4598 \end{bmatrix}, \quad \lambda_i(A_{o4}) = \begin{bmatrix} -0.0144 \\ -0.5100 \end{bmatrix}$$

$$\lambda_i(A_{o5}) = \begin{bmatrix} -0.1020 \\ 0.0691 \end{bmatrix}, \quad \lambda_i(A_{o6}) = \begin{bmatrix} -0.0095 \\ -0.5148 \end{bmatrix}$$

$$\lambda_i(A_{o7}) = \begin{bmatrix} -0.0164 \pm 0.1078 \end{bmatrix}, \quad \lambda_i(A_{o8}) = \begin{bmatrix} -0.0164 \pm 0.1321 \end{bmatrix}$$

### 5 Conclusion

The approach by controlled invariance of polyhedral sets seems to be particularly well fitted to the resolution of some disturbance rejection problems having complicating features such as control input constraints, and additional pole placement requirements. The main advantage of this approach is its computational efficiency which derives from the use of standard Linear Programming as the basic computational tool. Some extensions of the work presented in this study are currently under study, with the objective of decreasing the performance gap between simple solutions involving constant linear state feedback laws and dynamic or nonlinear $l^1$-optimal solutions.
References


