An Invariance Based Algorithm for the $l^1$ Optimal Control Problem

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Abstract

The purpose of this paper is to extend some results recently obtained on the $l^1$ optimal control problem using the controlled invariance approach. Feasibility of a given $l^1$ performance bound is tested by checking the non-emptiness of the supremal controlled invariant domain with stability contained in the admissible performance domain.

1 Introduction

Similarly to the $H_\infty$ optimal control problem, the $l^1$ optimal control problem seeks to minimize the influence of additive disturbances on the system outputs. In the $H_\infty$ control problem, this influence is measured in terms of the amplification of the disturbance input energy. In the $l^1$ control problem, this influence is measured in terms of the maximal possible amplification of the disturbance input magnitude.

A complete solution to the $l^1$ optimal control problem was given in [4]. Through the use of a parameterization of all stabilizing controllers, an optimal, possibly dynamic, linear controller is obtained by solving appropriate linear programs. In [11], it was shown that static non-linear controllers can provide the system with the same performance as linear dynamic ones. The same result was obtained in a constructive manner in [3] : a variable-structure controller was constructed by associating the achievement of an $l^1$ performance with the existence of a set of the state space which can be made invariant by state feedback.

The same idea was used in the paper [12]. But unlike [3] which relies on a linear dynamic controller to construct the static controller, it seeks to determine the best performance level for which there exists a corresponding controlled invariant set. A given performance level defines an admissible region in the state space. A necessary condition for the achievement of such a performance is that the largest controlled invariant set contained in the admissible region should not be empty. Construction of this set can thus be used to verify this condition. However, this condition is not sufficient in general, because the largest controlled invariant set may not be compact. In this case, controlled invariance does not guarantee asymptotic stability.

This problem can be tackled as in [12] by bounding in some way the admissible region defined by a given performance level. In this paper, it is proposed to solve this problem by computing the supremal controlled invariant set with stability contained in the admissible domain defined

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by the performance bound. It is the largest controlled invariant set in the given performance domain for which there exists a control sequence which asymptotically drives the state vector to the origin for the unforced system. Given a desired performance level, the computation of such a domain is performed in two steps:

1. Computation of the largest internally stabilizable domain for the unforced system.

2. Computation, for the system with additive disturbances, of the largest controlled invariant domain contained in the domain computed in step 1.

The advantage of this approach compared to that proposed in [12] is that it can directly treat the case of non-compact admissible domains.

2 Problem Formulation

Classically, the $l^1$ optimal control problem is the problem of optimally limiting the maximal amplitude of the output of a system subject to persistent bounded disturbances.

As in [4], [5], [3], [12], the case of full state feedback is considered. The considered discrete time systems are described by:

$$x_{k+1} = Ax_k + B_1 w_k + B_2 u_k,$$

$$z_k = C x_k,$$  \hfill (1)

where $x_k \in \mathbb{R}^n$ is the state vector, $w_k \in \mathbb{R}^q$ the disturbance input vector, $u_k \in \mathbb{R}^m$ the control input vector and $z_k \in \mathbb{R}^p$ the controlled output vector.

The disturbance vector $w_k$ is supposed to be restricted to a bounded set $D$:

$$w_k \in D.$$ \hfill (3)

In the multivariable case, the maximal signal amplitudes of the disturbance and the controlled output vectors are usually measured in terms the $l^\infty$-induced norm, defined for bounded sequences $h = \{h_k\}$, by:

$$\|h\|_{l^\infty} = \sup_{k \in \mathbb{N}} \|h_k\|_{\infty} < \infty.$$

The $l^\infty$ induced norm of system (1)-(3) is equal to $\gamma$ if

$$\sup_{\|w\|_{l^\infty} \leq 1} \|z\|_{l^\infty} = \gamma.$$ \hfill (4)

This implies for all $k \in \mathbb{N}$:

$$|Cx_k| \leq \gamma 1; \ \forall w = \{w_l\}; \ |w_l| \leq 1 \ \forall l \in \mathbb{N}$$ \hfill (5)

Under this formulation, the domain of admissible disturbance inputs is:

$$D = \{w_k; \ |w_k| \leq 1\}$$ \hfill (6)

and under an $l^1$ performance of $\gamma$, the domain of admissible state vectors is:

$$\Omega = \{x_k; \ |Cx_k| \leq \gamma 1\}.$$ \hfill (7)

As shown in [3] and in [12], the problems of existence and construction of a static state feedback achieving a given $l^1$ performance $\gamma$ rely on the property of controlled invariance for difference inclusions. This property can be analyzed either from viability theory [1] or from the geometric approach [13] (as in [7]). The objective of the $l^1$ optimal control problem is to find the minimal value of $\gamma$ for which there exists a control sequence achieving condition (5).
3 A geometric approach for the $l^1$ optimal control problem

3.1 Controlled invariant and internally stabilizable domains

**Definition 1** A domain $S \subset \mathbb{R}^n$ is controlled invariant with respect to system (1)-(3) if $\forall x \in S$ there exists a feasible control vector, $u \in \mathbb{R}^m$, such that: $Ax + B_1w + B_2u \in S$, $\forall w \in D$.

Note that this definition supposes that the disturbance vector is not measured.

**Definition 2** A domain $S \subset \mathbb{R}^n$ is internally stabilizable with respect to system (1) if, for the unforced system ($w_k = 0$), $\forall x_0 \in S$ there exists a control sequence $\{u_k\}$, $k = 0, 1, ..., i - 1$ such that $x_k \in \Omega$ and $x_k$ converges asymptotically to the origin.

It can be shown that both families of all controlled invariant domains and internally stabilizable domains are upper semilattices with respect to the operation “convex hull of the union”. This assures the existence in each family of a supremal member (a member which contains any other member):

$$C^\infty(\Omega, D) := \text{supremal controlled invariant domain contained in } \Omega.$$  
$$S^\infty(\Omega) := \text{supremal internally stabilizable domain contained in } \Omega.$$  

As for subspaces in the classical geometric theory [13], it is possible to combine controlled invariance and zero-controllability to define the concept of controlled invariance with internal stability.

**Definition 3** A domain $S \subset \mathbb{R}^n$ is controlled invariant with internal stability with respect to system (1), (3) if it is controlled invariant and internally stabilizable.

The family of all controlled invariant domains with internal stability included in a given domain $\Omega$ is also closed under the operation “convex hull of the union”. This implies the existence of the set:

$$S^\infty(\Omega, D) := \text{supremal controlled invariant domain with stability contained in } \Omega.$$  

Based on these facts and on the work of Shamma [12], the following result can be established:

**Proposition 1** There exists a controller which achieves a performance of $\gamma$ for system (1), (2) if and only if $S^\infty(\Omega, D)$ is not empty.

From this result, it can be seen that the solution of the $l^1$ optimal control problem is equivalent to finding the minimum value of $\gamma$ for which $S^\infty(\Omega, D)$ is not empty.

3.2 Characterization of $S^\infty(\Omega)$ and $S^\infty(\Omega, D)$

As a first step, the supremal internally stabilizable domain in $\Omega$, denoted $S^\infty(\Omega)$, is characterized.

**Proposition 2** Consider the following algorithm:

$S^0 = 0$  
$S^\mu = \{x \in \Omega; \exists u; Ax + B_2u \in S^{\mu-1}\}$, $\mu = 1, 2, ...$  

Then the supremal internally stabilizable domain contained in $\Omega$ is given by: $S^\infty(\Omega) = \lim_{\mu \to \infty} S^\mu$.  

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Proof: It suffices to note that $S^\mu$ is the set of states which can be driven to the origin in $\mu$ steps without leaving $\Omega$. Therefore, it is clear that $S^{\mu-1} \subset S^\mu$ and that the supremal internally stabilizable domain included in $\Omega$ is obtained when $\mu \to \infty$. □

The supremal controlled invariant domain with stability in $\Omega$ for system (1)-(3) can be characterized by:

**Proposition 3**

$$S^\infty(\Omega, D) = C^\infty(S^\infty(\Omega), D)$$

Proof: It follows directly from the Definition 3 of controlled invariance with internal stability. □

3.3 Computation of $S^\infty(\Omega)$

The supremal internally stabilizable domain contained in the domain of performance constraints (5) is $S^\infty(\Omega)$ with

$$\Omega = \{ x ; -\gamma 1 \leq Cx \leq \gamma 1 \}.$$

A simple case should firstly be pointed out: if the dimension of the output subspace, $p$ is strictly less than the number of independent controls, $m$, then, the algorithm in Proposition 2 generally admits the trivial solution $S^\infty(\Omega) = S^1 = \Omega$. It is not difficult to show, as in [8], that in this case the optimal value of $\gamma$ is: $\gamma^* = \|CB1\|_\infty$.

In the case $p = m$, system $(A, B_2, C)$ generally has invariant zeros. If these zeros are stable, they can be used as closed-loop poles, yielding for $S^\infty(\Omega)$ the associated zero directions as infinite directions. If the triplet $(A, B_2, C)$ also admits unstable invariant transmission zeros, computation of $S^\infty(\Omega)$ can be performed in the quotient space with respect to the stable zero directions subspace.

Hence, the general algorithm which will now be described may be either directly applied, in the case $p > m$, or applied to a reduced order system if $r = m$.

Let the set $S^\mu$ of the algorithm of Proposition 2 be given by:

$$S^\mu = \{ x ; G^\mu x \leq \rho^\mu \}.$$

The set $S^{\mu+1}$ is obtained from the iteration step (9) in the form:

$$S^{\mu+1} = \{ x ; \exists u; \begin{cases} G^\mu(Ax + B_2u) & \leq \rho^\mu \\ Cx & \leq \gamma 1 \\ -Cx & \leq \gamma 1 \end{cases} \}. \quad (10)$$

Application of the Fourier-Motzkin version of Farkas Lemma (see e.g. [10]) allows to eliminate variable $u$ in inequalities (10) by introducing a minimal generating set of the polyhedral cone:

$$\Gamma^\mu = \{ t \geq 0 ; \ tG^\mu B_2 = 0 \}.$$

The row vectors composing such a generating set form a non-negative matrix $T^\mu$. Then $S^{\mu+1}$ is given by:

$$S^{\mu+1} = \left\{ x ; \begin{bmatrix} T^\mu G^\mu A \\ C \\ -C \end{bmatrix} x \leq \begin{bmatrix} T^\mu \rho^\mu \\ \gamma 1 \\ \gamma 1 \end{bmatrix} \right\}. \quad (12)$$
After the removal of redundant constraints, the domain \( S^{\mu+1} \) is written in the form:
\[
S^{\mu+1} = \{ x; G^{\mu+1} x \leq \rho^{\mu+1} \}.
\]
The algorithm iterates until it has numerically reached a fixed point, which is a polyhedron denoted:
\[
S^F = \{ x; G^F x \leq \rho^F \}.
\]
This polyhedron may coincide with \( S^\infty \) when this domain is finitely generated, or with a finite approximation of the domain \( S^\infty \) when this domain is generated by an infinite number of constraints. In both cases, all the points of the constructed domain are internally stabilizable with respect to the unforced system.

### 3.4 Computation of \( S^\infty(\Omega, D) \)

From Proposition 3, the supremal controlled invariant domain with stability contained in the performance set \( \Omega \) can be computed as the supremal controlled invariant set contained in \( S^\infty(\Omega) \). Using the polyhedral (and possibly approximated) description \( S^F \) of \( S^\infty(\Omega) \), the problem reduces to the computation of the supremal controlled invariant set contained in a given polyhedron. Several algorithms have been proposed for such a computation [9], [2], [6], [12].

To construct the maximal controlled invariant domain in \( S^F \), it suffices then to construct the sequence of polyhedral sets
\[
C^\mu = \{ x ; R^\mu x \leq \eta^\mu \} \tag{13}
\]
with \( C^0 = S^F \).

As in [2], [7], the maximal one step admissible domain, \( Q(C^\mu) \), is defined as follows:
\[
Q(C^\mu) = \{ x ; \exists u ; R^\mu (Ax + B_1 w + B_2 u) \leq \eta^\mu, \forall w \in D \} \tag{14}
\]
The iteration formula is then: \( C^{\mu+1} = Q(C^\mu) \cap C^\mu \), which gives: \( C^\infty(S^F, D) = \lim_{\mu \to \infty} C^\mu \).

Using again the Fourier-Motzkin version of Farkas Lemma [10], vector \( u \) is eliminated in (14), using a non negative matrix \( T^\mu \) whose row-vectors constitute a minimal generating set of the polyhedral cone: \( \{ t \geq 0 ; t R^\mu B_2 = 0 \} \).

The set \( Q(C^\mu) \) thus coincides with the polyhedral domain:
\[
\{ x ; T^\mu R^\mu A \leq T^\mu (\eta^\mu - \delta^\mu) \}
\]
where \( \delta^\mu_i = \max_{w \in D} R^\mu_i B_1 w = |R^\mu_i B_1|_1 \). Thus,
\[
C^{\mu+1} = \left\{ x ; \left[ \begin{array}{c} R^\mu \\ T^\mu R^\mu A \end{array} \right] x \leq \left[ \begin{array}{c} \eta^\mu \\ T^\mu (\eta^\mu - |R^\mu B_1|_1) \end{array} \right] \right\}.
\]

It is clear from this construction that \( C^\infty(S^F, D) = Q(C^\infty(S^F, D)) \subset S^F \).

Again, this is a fixed point algorithm for which numerical convergence to a finitely generated polyhedron is achieved.

### 3.5 Computation of the optimal achievable performance \( \gamma^* \)

The iterative computation of \( C^\infty(S^F, D) \) with the set \( S^F \) computed for decreasing values of the \( l^1 \) performance index, \( \gamma \), eventually leads to the minimal achievable value, \( \gamma^* \). For any value of \( \gamma < \gamma^* \), the set \( C^\infty(S^F, D) \) is the empty set.

Such an iterative construction may be rather time consuming. The following result shows that the computation of the set \( S^F = \{ x; G^F x \leq \rho^F \} \) only needs to be performed for one positive value of \( \gamma \).
Proposition 4  Consider the non-empty set \( \Omega' = \lambda \Omega \), with \( \lambda > 0 \). Then \( S^\infty(\Omega') = \lambda S^\infty(\Omega) \).

Proof: Consider the algorithm described in Proposition 2, and suppose \( S'^{\mu - 1} = \lambda S^{\mu - 1} \). Then, \( S'^{\mu} = \{ x' \in \lambda \Omega, \exists u'; Ax' + B_2 u' \in \lambda S^{\mu - 1} \} = \{ \lambda x, x \in \Omega, \exists u; \lambda (Ax + B_2 u) \in \lambda S^{\mu - 1} \} = \lambda S^\mu \).

The Proposition follows by induction, since it is clear that \( S'^{1} = \lambda S^{1} \). \( \square \)

Let \( \Omega' = \{ x; |Cx| \leq \gamma' \} \) be the admissible performance domain associated with \( \gamma' > 0 \). From the preceding Proposition, the set \( S^\infty(\Omega') \) is homothetic to \( S^\infty(\Omega) \) with rate \( \frac{\gamma}{\gamma'} \). Therefore, a polyhedral approximation \( S^F(\Omega') \) of the supremal internally stabilizable domain contained in \( \Omega' \) is:

\[
S^F(\Omega') = \{ x; G^Fx \leq \frac{\gamma'}{\gamma} \rho^F \}.
\]

3.6  Example

The proposed algorithm has been applied to the numerical example 6.2 presented in [12]. The fact that no truncation was a priori imposed on the admissible performance domain has allowed the construction of a non-empty controlled invariant domain \( S^\infty(\Omega, D) \) for values of \( \gamma \) smaller than the one obtained in [12].

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4.6 & -23.5 & 2.7 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 1.51 & -2.5 & 1 \end{bmatrix}.
\]

With \( \Omega = S(C, \gamma 1) \), \( S^\infty(\Omega) = S(G^F, \gamma \rho^F) \), where

\[
G^F = \begin{bmatrix} 0 & 0 & 1.0209 \\ 0 & -1.51 & 2.5000 \\ 0 & -1.51 & 1.8960 \\ 0 & -1.51 & 1.7036 \\ 0 & -1.51 & 1.6136 \\ 0 & -1.51 & 1.5642 \\ 0 & -1.51 & 1.5347 \\ 0 & -1.51 & 1.5161 \\ 0 & -1.51 & 1.5040 \\ 0 & -1.51 & 1.4960 \\ 0 & -1.51 & 1.4870 \\ 0 & -1.51 & 1.4791 \\ 1.51 & -2.5 & 1 \end{bmatrix}, \quad \rho^F = \begin{bmatrix} 102.0824 \\ 100.9952 \\ 41.3981 \\ 22.8344 \\ 14.4037 \\ 9.9262 \\ 7.3458 \\ 5.7866 \\ 4.8168 \\ 4.2027 \\ 3.5554 \\ 3.0871 \\ 1.0000 \end{bmatrix}.
\]
With $\gamma = 3.06$ and a precision of $10^{-3}$, $S^\infty(\Omega, D) = S(R^F, \gamma 1)$, with

$$R^F = \begin{bmatrix}
1.51 & -2.5 & 1 \\
0 & 1.4575 & -1.4353 \\
0 & 1.4879 & -1.4741 \\
0 & 1.5168 & -1.5416 \\
0 & 1.5174 & -1.5235 \\
0 & 1.5058 & -1.4998 \\
0 & 1.4059 & -1.5024 \\
0 & 1.4921 & -1.5457 \\
0 & -0.0000 & 0.3336 \\
0 & 0.4511 & -0.7468 \\
0 & 0.8959 & -1.1249 \\
0 & 1.2223 & -1.3790
\end{bmatrix}.$$

4 Conclusion

This paper has described an algorithm for solving the $l^1$ optimal control problem. This algorithm treats the general case when the performance domain is not supposed compact in the state space. The proposed technique both tackles the stability problem and the performance requirement problem by constructing the supremal controlled invariant domain with stability contained in the supremal internally stabilizable domain in the performance set.

References


