GENERALIZED MINIMUM VARIANCE
CONTROL OF CONSTRAINED MULTI-VARIABLE SYSTEMS

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Abstract

We propose a control scheme for a discrete-time linear stochastic multi-input multi-output (MIMO) system, with imperfectly known or slowly time-varying parameters.

The selected approach is generalized minimum variance control, but the method can easily be extended to generalized predictive control. The quadratic criterion to be minimized penalizes the variance of the difference between the output of a reference model and the predicted variable to be regulated. It also penalizes the difference between two successive control vectors. We have also included, in the model, quadratic constraints on control variables. In application to production-planning problems, these correspond to minimal and maximal production capacities.

The predicted auxiliary output is determined by an extended least-squares algorithm, and the optimality conditions evolve by periodical resetting of the criterion weights. Weighting parameters can therefore be periodically adjusted by a Robbins-Monro type algorithm to solve the system of equations obtained from Kuhn-Tucker necessary conditions.

Key words
Adaptive control, MIMO systems, production planning.

1. INTRODUCTION

Control of stochastic linear systems with unknown or slowly time-varying parameters is of great theoretical and practical interest. The two main approaches in this field, model reference and self-tuning, have both contributed to the emergence of generalized predictive adaptive control, initially formulated for single input single-output (SISO) systems by Clarke and Gawthrop [1] and next extended to the multi-input multi-output (MIMO) case by Koivo [2], Favier and Hassani [3], Irving [4], and others. Through this control scheme, a trade-off can be found between minimizing the output error variance and smoothing control variations.

In this paper, the selected one-step quadratic cost function compromises between the deviation from the reference model output and the distance of the current control vector to the preceding one. The proposed control scheme performs on-line adjustment of the weighting parameters of the criterion, so that the control vector always stays in the feasible region generated by a set of quadratic inequality constraints.

This scheme has been applied to production planning problems with minimal and maximal production capacities. The selected objective function drives the system toward anticipated future demands while avoiding strong variations of production levels.

2. THE CONSTRAINED CONTROL PROBLEM FORMULATION

We consider stochastic discrete-time linear systems described by the following vector equation:

\[ A(z^{-1})y_t = B(z^{-1})u_{t-k} + C(z^{-1})e_t + d \]  

where \( y_t \), \( u_t \), and \( e_t \) are the output vector at time \( t \), the control vector at time \( t \), and a sequence of uncorrelated zero-mean random vectors with constant covariance matrices. \( A(0) \), \( B(z^{-1}) \), and \( C(z^{-1}) \) are polynomial matrices with respective dimensions \( nxn \), \( nxm \), and \( nxn \). \( z^{-1} \) is the backward-shift operator. \( A(z^{-1}) \), \( B(z^{-1}) \), and \( C(z^{-1}) \) are assumed to lie inside the unit circle.

The system control vector is subject to constraints of the following type:

\[ \mathbf{g}_i^T \mathbf{u}_t \leq \mathbf{p}_i, \quad i = 1, \ldots, p \]

with \( \mathbf{g}_i \in \mathbb{R}^m \), \( \mathbf{p}_i \in \mathbb{R}^p \); \( p \) is the number of constraints.

The system is subject to constraints on the control vector. In most practical cases, the constraints are linear symmetrical of the following type:

\[ \mathbf{g}_i^T \mathbf{u}_t \leq \mathbf{p}_i, \quad i = 1, \ldots, p \]

with \( \mathbf{g}_i \in \mathbb{R}^m \), \( \mathbf{p}_i \in \mathbb{R}^p \); \( p \) is the number of constraints.

In some production-system models [5], each component \( u_i(t) \) of the control \( u_t \) represents the production level of activity \( i \) during period \( t \) and is subject to production capacity constraints of the following type:

\[ 0 \leq u_i(t) \leq c_i \]
Such constraints can be made symmetrical by translating $u_t$.

Constraints (3) can also be written
\[ (G_i^T u_t)^2 < \rho_i^2 \]
And a more general form of quadratic constraints is
\[ \|G_i u_t\|^2 < \rho_i^2 \]
for $i = 1, \ldots, p$
where $G_i$ usually represents a line-vector of $R^n$ but may also represent a matrix of $R^{n \times m}$ with $q$ being a positive integer.

3. DERIVATION OF THE PREDICTIVE CONTROLLER

In order to solve the optimization problem, we need the k-step prediction of the auxiliary output defined by
\[ \hat{\psi}_{t+k} = \Lambda_{\psi}(z^{-1}) Y_{t+k} \]
We denote $\hat{\psi}_{t+k/t}$ as the optimal estimate of $\psi_{t+k}$ at time $t$. We get
\[ \hat{\psi}_{t+k} - \hat{\psi}_{t+k/t} + \varepsilon_{t+k} \]
$\varepsilon_{t+k}$ is the prediction error. It will be shown later that this error is uncorrelated to $\hat{\psi}_{t+k/t}$

Therefore, criterion (2) can be written for time $t$:
\[ J = \left( \| \hat{\psi}_{t+k/t} - \hat{\psi}_{t+k} \|^2 + I G_i (u_t - u_{t-1}) \|^2 \right)^2 + \text{trace} \ E(\varepsilon_{t+k} \varepsilon_{t+k}^T) \]

The problem then reduces to a deterministic optimization problem under inequality constraints. Optimality conditions can be obtained from the Lagrangean formulation of the problem:
\[ J_t = \left( \| \hat{\psi}_{t+k/t} - \hat{\psi}_{t+k} \|^2 + I G_i (u_t - u_{t-1}) \|^2 \right)^2 + \sum_{i=1}^{p} \lambda_i(t) \left( G_i^T u_t^2 - \rho_i^2 \right) + \text{trace} \ E(\varepsilon_{t+k} \varepsilon_{t+k}^T) \]
with $\lambda_i(t) > 0$ for $i = 1, \ldots, p$.

Lagrange multipliers play the role of weighting coefficients on control variables. They must be adjusted so as to satisfy Kuhn-Tucker necessary conditions:
\[ \lambda_i(t) \left( G_i^T u_t^2 - \rho_i^2 \right) = 0 \quad \text{for } i = 1, \ldots, p \]

The optimal control law can be derived from
\[ \frac{\delta J}{\delta u_t} = 0 \]
that is,
\[ \left[ \frac{\delta \hat{\psi}_{t+k/t}}{\delta u_t} \right]^T \left( \hat{\psi}_{t+k/t} - \hat{\psi}_{t+k} \right)^T + Q_i(z^{-1}) u_t = 0 \]
with $Q_i(z^{-1}) = (Q_{i}^T Q_i + \sum_{i=1}^{p} \lambda_i(t) G_i^T G_i) - (Q_{i}^T Q_i) z^{-1}$

Matrix $Q_i(z^{-1})$ can be considered as the new control weighting matrix in the criterion.

The predicted auxiliary output can be obtained from the following Diophantine equations:
\[ C(z^{-1}) A_{\psi}(z^{-1}) = A(z^{-1}) E(z^{-1}) + z^{-k} F(z^{-1}) \]
The couple of $n \times n$ polynomial matrices $[E(z^{-1}), F(z^{-1})]$ is unique under conditions $\text{deg } E(z^{-1}) = k-1$ and $E(0) = I$.

Let us also introduce, as do Borison [6] and Kucera [7], the couple of $n \times n$ polynomial matrices $[\tilde{E}(z^{-1}), \tilde{F}(z^{-1})]$ such that
\[ \tilde{E}(z^{-1}) \tilde{F}(z^{-1}) - \tilde{F}(z^{-1}) \tilde{E}(z^{-1}) = \tilde{E}(z^{-1}) \]
with $\det \tilde{E}(z^{-1}) = \det E(z^{-1})$ and $\tilde{E}(0) = I$.

The couple $[\tilde{E}(z^{-1}), \tilde{F}(z^{-1})]$ always exists but it is not necessarily unique. It verifies the following relation:
\[ \tilde{C}(z^{-1}) A_{\psi}(z^{-1}) = \tilde{E}(z^{-1}) A(z^{-1}) + z^{-k} \tilde{F}(z^{-1}) \]
with $\tilde{C}(z^{-1})$, a polynomial matrix defined by
\[ \tilde{C}(z^{-1}) = E(z^{-1}) - \tilde{E}(z^{-1}) C(z^{-1}) \]
and $\det \tilde{C}(z^{-1}) = \det C(z^{-1})$.

The requirement that $A_{\psi}(z^{-1})$ be a polynomial and not a polynomial matrix has been used to commute $A_{\psi}(z^{-1})$ and $E(z^{-1})$ to establish relation (11).

By left-multiplication of equation (1) by $E(z^{-1})$ and use of relations (5), (11), and (12), we get
\[ \tilde{C}(z^{-1}) \tilde{\psi}_{t+k} = \tilde{F}(z^{-1}) Y_t + \tilde{E}(z^{-1}) B(z^{-1}) u_t + \gamma + \tilde{C}(z^{-1}) E(z^{-1}) e_{t+k} \]
with $\gamma = E(1) d$.

We can then obtain the predicted auxiliary output $\hat{\psi}_{t+k/t}$ from the following relation:
\[ \tilde{C}(z^{-1}) \tilde{\psi}_{t+k/t} - \tilde{F}(z^{-1}) Y_t + \tilde{E}(z^{-1}) B(z^{-1}) u_t + \gamma \]
The k-step prediction error $\varepsilon_{t+k}$ can be written
\[ \varepsilon_{t+k} = E(z^{-1}) e_{t+k} - e_{t+k} + E_{t+k-1} e_{t+k-1} + \ldots + E_{t-1} e_{t+1} \]
This expression shows that $\varepsilon_{t+k}$ and $\hat{\psi}_{t+k/t}$ are independent.

Using the following notations:
\[ \tilde{E}(z^{-1}) B(z^{-1}) = (\tilde{EB})_b + (\tilde{EB})_1 z^{-1} + \ldots \]
we obtain from equation (13)
\[ \left[ \frac{\delta \hat{\psi}_{t+k/t}}{\delta u_t} \right]^T = (\tilde{EB})_b^T \]
and with $\tilde{E}(0) = I$
\[ \left[ \frac{\delta \hat{\psi}_{t+k/t}}{\delta u_t} \right]^T = E_b^T \]
Then condition (8) takes the form
\[ R_b^T \left( \Psi_{t+k/t} - A_b(z^{-1}) \Psi_{t+k} \right) + Q_t(z^{-1}) u_t = 0 \] (14)

Equation (14) can be directly used to compute the control vector \( u_t \). However, if \( B_b \) has dimension nxn and is non-singular, the generalized output of the system can be defined by
\[ G_{t+k} = \Psi_{t+k} - A_b(z^{-1}) \Psi_{t+k} + (B_b)\text{'}^{-1} Q_t(z^{-1}) u_t \]

Its prediction at time t is
\[ \hat{G}_{t+k/t} = \hat{\Psi}_{t+k/t} - A_b(z^{-1}) \hat{\Psi}_{t+k} + (B_b)\text{'}^{-1} Q_t(z^{-1}) u_t \]

Condition (14) then implies the classical result in minimum variance control: The control vector can be obtained by setting \( \hat{G}_{t+k/t} = 0 \). In our particular case, this result has been obtained with a modified control weighting matrix.

4. SELF-TUNING IMPLEMENTATION

Adaptive control has now become a fully recognized field of study and has been applied to many practical problems. Some applications on specific industrial-production processes have been reported [8, 9]. Here the model equation is of a general type, but the particular field of application that has been considered is medium-term production planning problems. In this context, each component \( y_i(t) \) of the output vector represents the amount of good i available for sale at time t. This amount has been previously produced by different activities at different times and with different efficiencies. Industrial production processes are not constant in time. They may vary with running conditions of the machines and with availability of resources and production factors. Therefore, production plans should be adjusted according to the real measured output (closed-loop control) and according to evolutions of production parameters, which can only be indirectly detected through recursive identification.

Self-tuning implementation of the generalized minimum variance control law can be achieved through an implicit identification scheme.

Equation (13) can be written as
\[ \hat{\Psi}_{t+k/t} = \Theta \cdot x_t \] (15)

with \( \Theta = \left[ B_b, (EB)_1, \ldots, (EB)_P, \ldots, (EB)_Y, (EB)_C, \ldots \right] \)

and \( x_t = \left[ u_t, u_{t-1}, \ldots, u_{t-1}, \ldots, 1, \hat{\Psi}_{t+k-1/t}, \ldots \right] \)

Let \( \Theta_i \) be the i-th line of \( \Theta \).

At time t, vectors \( \hat{\Psi}_{t+k/t} \) appearing in the expression of \( x_t \) with \( j-k-1, \ldots, 1 \) are not directly known. They can be estimated after resolution of Diophantine equations of the same type as (11) [2].

\[ C_j(z^{-1}) \hat{\Psi}_{t+j/t} - F_j(z^{-1}) \hat{y}_t + E_j(z^{-1}) B(z^{-1}) u_t + \gamma_j \]

with \( \gamma_j = \hat{E}_j(1).d \).

In practice, as Gawthrop [10] has explained, we can replace, in the expression of \( x_t \), \( \Psi_{t+k-p/t} \) by \( \hat{\Psi}_{t+k-p/t} \) for \( p=1, \ldots, k-1 \) and \( t \geq p \).

Matrix \( \Theta \) can be identified by a recursive least-squares algorithm:
\[ \Theta_{t+1} = \frac{1}{\beta} \Theta_t - \frac{1}{\beta} (K_t \Theta_t - K_t \Theta_t) \]

(16)

Matrix \( B \) is the forgetting factor, usually chosen as between 0.9 and 1.

Since vector \( x_t \) actually includes approximations on predicted auxiliary outputs, algorithm (16) can be considered as an extended recursive least-squares algorithm (ERLS). However, in the case of fixed or slowly varying parameters, the control law still converges to the generalized minimum variance control law by virtue of the self-tuning property, initially demonstrated by Aström and Wittenmark [11] in the case of minimum variance control and extended to the case of generalized predictive control by Welstead, Prager, and Zanker [12].

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\[ C_j(z^{-1}) \hat{\Psi}_{t+j/t} - F_j(z^{-1}) \hat{y}_t + E_j(z^{-1}) B(z^{-1}) u_t + \gamma_j \]

for \( j=1, \ldots, k-1 \).

It yields
\( \gamma(s) \) is a sequence of positive scalars smaller than 1 so that \( \lambda_i^*(t) \) remains positive, and \( \gamma(s) \to 0 \) for \( s \to \infty \).

At each time \( t \), the following steps have to be performed:

1. Measure \( y_t \) and read \( y_{t+1}^* \)
2. Compute \( \psi_t = A(z^{-1}) y_t \)
3. Use the ERLS algorithm to estimate \( \hat{\Theta} \)
4. Form the data vector \( x_t \)
5. Initialize \( \lambda_i(t) = \lambda_i^* \)
6. Compute \( Q_t(0) \) and \( u_t \) by expression (18)
7. Check if \( |\lambda_i^*(t)\{u_t^2 + (G_j u_t)^2\}| \leq 6, 6 \not\in 1 \) and \( \delta > 0 \), for \( i = 1, \ldots, p \)

If all these constraints are satisfied with sufficient precision, keep \( u_t \) as the control vector, do \( t = t + 1 \), and go to step 1; if not, go to step 8.

8. For all the violated constraints, update \( \lambda_i(t) \) by

\[
\lambda_i^{t+1}(t) = \lambda_i^*(t) + \frac{\gamma(s)}{\rho_i} \lambda_i^*(t) \{u_t^2 + (G_j u_t)^2\}
\]

and go to step 6.

Constraints on the input vector may in some cases imply large deviations of the output vector from the reference model output, and the system may even fail to converge. Another possible cause of divergence is structural, when the closed-loop system determinant has its zeros outside the unit circle. When \( B_i \) is an nonsingular, this determinant can be written as

\[
\det \begin{bmatrix} A(z^{-1}) & B(z^{-1}) \\ z^{-k} F(z^{-1}) & E(z^{-1}) B(z^{-1}) + C(z^{-1}) (B_i^{-1}) Q_i(z^{-1}) \end{bmatrix}
\]

In the case of production planning, the risk of divergence is not as serious as it is for the control of fast processes, since the decision-maker can easily interface in the process. If the value of the cost function gets too high, it may become necessary to change the production scheme or to increase production capacities.

5. SIMULATION

The control algorithm that has been presented will now be illustrated on a very simple production process. It includes three activities, producing two types of goods. Each component \( u_i(t) \) of the control vector at time \( t \) represents the running level of activity \( i \) during the time interval \( (t, t+1) \), and each component \( y_j(t) \) of the output vector at time \( t \) represents the available stock of product \( j \) at time \( t \).

The discrete-time stochastic linear model of the system is

\[
A(z^{-1}) y_{t+1} = B(z^{-1}) u_t + C(z^{-1}) e_{t+1}
\]

with

\[
\begin{align*}
A(z^{-1}) &= (1-z^{-1}) . 1 \\
B(z^{-1}) &= (2 0 1) z^{-1} + (0.5 -0.5 0.75) \\
C(z^{-1}) &= 1
\end{align*}
\]

The polynomial matrix \( A(z^{-1}) = (1-z^{-1}) . 1 \) characterizes stock conservation of non-depreciable goods. Equation (19) is an input-output representation of the production system, which can be derived from the production balance sheets at every activity station.

This type of model can be graphically described by a pseudo-Petri net (Fig. 1) with stocks represented by places and activities by transitions [5].

Figure 1. Representation of a production process by a pseudo-Petri net.

Production levels are subject to the following capacity constraints:

\[
\begin{align*}
0 &\leq u_1(t) < 8 & (20) \\
0 &\leq u_2(t) < 6 & (21) \\
0 &\leq u_3(t) < 6 & (22) \\
5 &\leq u_1(t) + u_2(t) + u_3(t) < 15 & (23)
\end{align*}
\]

for \( t = 0,1, \ldots \).

In order to get symmetrical constraints, we define a translated control vector:

\[
\tilde{u}_t = u_t + \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}
\]

Equation (19) can be written as

\[
A(z^{-1}) \tilde{y}_{t+1} = B(z^{-1}) \tilde{u}_t + e_{t+1} + d
\]

with \( d = \begin{bmatrix} 3.75 \\ 1 \end{bmatrix} \)

Under constraints:

\[
\begin{align*}
(G_1 . \tilde{u}_t)^2 &\leq 16 \quad \text{with } G_1 = (1 0 0) \\
(G_2 . \tilde{u}_t)^2 &\leq 9 \quad \text{with } G_2 = (0 1 0) \\
(G_3 . \tilde{u}_t)^2 &\leq 9 \quad \text{with } G_3 = (0 0 1) \\
(G_4 . \tilde{u}_t)^2 &\leq 25 \quad \text{with } G_4 = (1 1 1)
\end{align*}
\]

We have selected \( Q_0 = I, A_0(z^{-1}) = 1 - 0.75 z^{-1} \).

The set point \( \tilde{y}_t^* \) is a step function, as depicted in Fig. 2.

The model structure, and particularly the delays in the system, are assumed to be known but the values of the parameters are assumed to be unknown. The RLs algorithm (17) is used to estimate the coefficients of matrix \( \Theta \) such that

\[
\psi_{t+1/t} = \Theta x_t
\]

with \( x_t = (u_t^2, u_{t-1}^2, y_t^2, 1) \).
A stopping test is used when the innovation vector gets too small.

We use the following parameters and initial values:

\[ \beta = 0.95 \]
\[ \lambda^0 = 0.1 \]
\[ \Theta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ P_0 = 1000 \times \mathbb{I} \]

The simulation results of Fig. 2 show how the system output approaches the set point with the desired dynamics. In Figs. 3 and 4, we can check that the constraints are always satisfied, as a result of the evolution of Lagrange parameters represented in Figs. 5 and 6.

Several methods have been proposed to overcome the major difficulty in implementing self-tuning algorithms, that is, the risk of generating control efforts that are too large. In Clarke and Gawthrop's generalized predictive scheme [1], a compromise can be found between approaching the target and keeping the control effort within acceptable bounds, but the choice of the weighting parameters is both delicate and somewhat arbitrary. Actually, the objective function is mainly a tool for coming up to the desired behaviour of the system. In this view, an on-line adjustment of the weighting parameters can be a judicious way of efficiently driving the system all along its trajectory. The quadratic formulation of the constraints that has been used in this paper allows a direct and rational modification of the weighting parameters in the criterion. The control scheme relies on the optimization of a one-step Lagrangean function, and the respect of Kuhn-Tucker conditions guarantees that input constraints are not violated.

**REFERENCES**


