(A, B)-invariant Polyhedral Sets of Linear Discrete-time Systems

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Abstract

The problem of confining the trajectory of a linear discrete-time system in a given polyhedral domain is addressed through the concept of (A, B)-invariance. Firstly, an explicit characterization of (A, B)-invariance of convex polyhedra is proposed. Such a characterization amounts to necessary and sufficient conditions in the form of linear matrix relations, and presents two major advantages compared to the ones found in the literature: it applies to any convex polyhedron and does not require the computation of vertices. Such advantages are particularly felt in the computation of the supremal (A, B)-invariant set included in a given polyhedron, for which a numerical method is proposed. The problem of computing a control law which forces the system trajectories to evolve inside an (A, B)-invariant polyhedron is treated as well. The (A, B)-invariance relations are then generalized to systems subject to linear constraints on the control vector, and to persistently disturbed systems.

1 Introduction

Linear systems subject to point-wise in time constraints have proved to be objects of great interest both for theoreticians in optimization and control and for practitioners. The usefulness of this model is largely due to the fact that in real-life control problems, such constraints can arise either from physical limitations inherent in a process of linear behavior, or from the validity domain of the linearization of a non-linear system.

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In particular, the so-called **positive invariance** approach has been used to solve a large number of problems on constrained dynamical systems. A set in the state space is **positively invariant** if any trajectory originated from this set does not leave it. It is a fact that physical limitations inherent to the operation of actual dynamical systems very often result in linear constraints on the state and/or control variables. As a consequence, much effort was directed towards the development of the theory of positively invariant polyhedral sets, initially for discrete-time systems [4, 2, 20, 6], but also for continuous-time systems [5, 6, 9].

The most direct application of this theory to the resolution of constrained control problems consists in verifying the existence of a state feedback control law which achieves positive invariance of the polyhedron defined by the constraints. The drawbacks of this approach are twofold: closed-loop positive invariance of the polyhedron of constraints can seldom be achieved; the use of linear state-feedback can restrict the possibility of achieving constraints satisfaction. One is then led to consider other sets different from the set of constraints, and also more general control laws. Such considerations are naturally embedded in the concept of \((A, B)\)-invariance.

In the framework of the geometric approach to the control of linear systems, the concept of \((A, B)\)-invariance of subspaces plays an important role. In particular, it has been widely applied to the solution of some classical control problems, such as disturbance and input/output decoupling [24, 1]. This concept of \((A, B)\)-invariance, with a different denomination though, has also been applied to convex polyhedra, to characterize the possibility of controlling discrete-time systems subject to point-wise in time trajectory constraints. Seminal works on this subject, in the sixties and seventies (see e.g. [23, 17, 3]) have treated this problem essentially at the conceptual level. Concerning more applied results, a vertex-by-vertex characterization has been proposed in [18] for compact polyhedra. In [21] this approach was used to solve a minimum time control problem. Finally, in [7] the results of [18] were extended to uncertain additively disturbed systems. Two major drawbacks can however be detected in this approach. Firstly, it only applies to compact polyhedra. In several problems the polyhedron defined by state, control or output constraints is not necessarily compact. Secondly, depending on the complexity of the considered polyhedron, the computation of its vertices, and consequently the test of \((A, B)\)-invariance, can become numerically very expensive.

The polyhedra defined by linear constraints are not \((A, B)\)-invariant in general. Constraints satisfaction can however be achieved if the initial states are forced to belong to an \((A, B)\)-invariant set contained in the set of constraints. The set which is generally chosen, because it is the most easily characterized and computed, is the supremal set (or an approximation of it), that is the set which contains all the other sets. Some algorithms for computation of this set have been proposed [19, 7]. Because they are based on the vertex-by-vertex characterization of \((A, B)\)-invariance, these algorithms present also considerable numerical drawbacks.

The main contribution of this paper is in characterizing the property of \((A, B)\)-invariance of convex polyhedra for discrete-time systems. By application of Farkas' Lemma, necessary and sufficient conditions under which a general convex polyhedron is \((A, B)\)-invariant are established in the form of linear matrix relations. A particular form of such relations is derived in the case of 0-symmetrical polyhedra. The problem of computing a control law which achieves closed-loop positive invariance of an \((A, B)\)-invariant polyhedron is treated as well. A piecewise linear control law, which is an extension to the non-compact case of that proposed in [18, 7], is derived to the
general case. The \((A, B)\)-contractive polyhedral sets are also introduced and characterized. Then, the supremal \((A, B)\)-invariant sets contained in a given polyhedron are studied. Such sets are theoretically characterized and a numerical method, based on the \((A, B)\)-invariance relations, is proposed for their computation.

**Notation:** In mathematical expressions, the symbol "\(" stands for "such that". By convention, inequalities between vectors and inequalities between matrices are component-wise. \(\mathbb{N}\) and \(\mathbb{R}\) represent respectively the sets of natural and real numbers. \(I_n\) represents the identity matrix of order \(n\). A vector (or matrix) is said to be non-negative if all its components are non-negative. The absolute value \(|M|\) (resp. \(|v|\)) of a matrix \(M\) (resp. of a vector \(v\)) is defined as the matrix (resp. vector) of the absolute value of its components. \(M_i\) represents the \(i\)-th row of \(M\), and \(M_{ij}\) represents the element of row \(i\) and column \(j\) of matrix \(M\).

## 2 Preliminaries

Let us firstly recall some fundamental concepts related to linear spaces and polyhedral sets.

Let \(\Omega\) be a set in a normed linear space \(\mathcal{X}\), with the norm represented by \(|\cdot|\). The set \(\Omega\) is said to be **bounded** if there exists a scalar \(s > 0\) such that \(|x| \leq s\), \(\forall x \in \Omega\). \(\Omega\) is **closed** if it contains all its closure points. Finally, \(\Omega\) is **compact** if it is bounded and closed.

In this work, only closed set are studied, and the considered linear spaces are over the field of real numbers, \(\mathbb{R}\).

### 2.1 Polyhedral sets

A closed convex polyhedron of \(\mathbb{R}^n\), \(R[G, \rho]\), with \(G \in \mathbb{R}^{g \times n}\) and \(\rho \in \mathbb{R}^g\), is defined by the set of linear inequalities:

\[
R[G, \rho] = \{ x \in \mathbb{R}^n : Gx \leq \rho \}.
\]

A \(0\)-symmetrical convex polyhedron of \(\mathbb{R}^n\), \(S(Q, \mu)\), with \(\mu \geq 0\), is defined by the set of linear inequalities:

\[
S(Q, \mu) = \{ x \in \mathbb{R}^n : |Qx| \leq \mu \}.
\]

A **polytope** is a compact polyhedron.

A polyhedral cone of \(\mathbb{R}^n\), \(R[G, 0]\), is defined by the set of linear inequalities:

\[
R[G, 0] = \{ x \in \mathbb{R}^n : Gx \leq 0 \}.
\]

The column vectors of matrix \(M\) form a **generating set** of the polyhedral cone \(R[G, 0]\) if and only if there exists a nonnegative vector \(\xi\) such that \(x = M\xi\), \(\forall x \in R[G, 0]\). Each column vector of \(M\) is then called a **generator** of \(R[G, 0]\). A generating set of \(R[G, 0]\) is said to be a **minimal generating set** if it is defined by the smallest number of generators.
The affine hull of \( R[G, \rho] \) is defined by \( \mathcal{A} = R[G, 0] \cap -R[G, 0] = \{ x \in \mathbb{R}^n : Gx = 0 \} \) [22]. A polyhedral cone \( R[G, 0] \) can be decomposed into the form \( R[G, 0] = P + \mathcal{A} \), where \( P \) is a proper cone. If \( \mathcal{A} = \{0\} \), the cone is said to be pointed and a minimal generating set can be obtained by selecting a nonzero vector from each extremal ray of the cone [10].

### 2.2 Farkas’ Lemma

**Lemma 2.1** Let \( M \) be a matrix and \( v \) a vector. Then \( \exists x : Mx \leq v \) if and only if \( yv \geq 0 \forall y \geq 0 : yM = 0 \).

The set of row vectors \( y \) such that \( y \geq 0 \), \( yM = 0 \) form a pointed polyhedral cone. From now on, this cone will be called nonnegative left kernel of matrix \( M \).

Let \( W \) be a nonnegative matrix whose row vectors form a generating set of the nonnegative left kernel of \( M \). Then, lemma (2.1) can be re-stated as follows:

**Lemma 2.2** \( \exists x : Mx \leq v \) if and only if \( Wv \geq 0 \).

As shown in [21], it is possible to compute the matrix \( W \) by means of Fourier-Motzkin elimination technique [22].

The following variation of Farkas’ Lemma gives a set of necessary and sufficient conditions under which \( R[G, \rho] \subset R[P, \psi] \), with \( P \in \mathbb{R}^{n \times n} \) and \( \psi \in \mathbb{R}^p \) [20] :

**Lemma 2.3** For \( x \in \mathbb{R}^n \), the set of inequalities \( Px \leq \psi \) is verified by each point of the nonempty convex polyhedron defined by \( Gx \leq \rho \) if and only if there exists a (dual) non-negative matrix \( U \in \mathbb{R}^{p \times n} \) such that:

\[
UG = P, \\
U\rho \leq \psi.
\]

### 3 Characterization of \((A, B)\)-invariance

Consider the linear time-invariant discrete-time system described by:

\[
x(k+1) = Ax(k) + Bu(k),
\]

with \( k \in \mathcal{N} \). \( x \in \mathbb{R}^n \) is the state vector and \( u \in \mathbb{R}^m \) is the control vector.

Let us firstly recall the concept of positive invariance.

**Definition 3.1** A non-empty closed set \( \Omega \subset \mathbb{R}^n \) is said to be **positively invariant** with respect to a dynamical system \( x(k+1) = f(x(k)) \) if \( \forall x(0) \in \Omega, x(k) \in \Omega, \forall k \in \mathcal{N} \).
An \((A, B)\)-invariant set is defined as follows:

**Definition 3.2** A non-empty closed set \(\Omega \subset \mathbb{R}^n\) is said to be \((A, B)\)-invariant with respect to system (1) if \(\forall x \in \Omega\) there exists a control vector \(u \in \mathbb{R}^m\) such that \(Ax + Bu \in \Omega\).

One can also say that \(\Omega\) is \((A, B)\)-invariant if \(\forall x(0) \in \Omega\) there exists a control sequence \(\{u(k)\}, k \in \mathcal{N}\) such that the trajectory of the state vector of the controlled system is completely contained in \(\Omega\). In other words, the state \(x(k)\) can be forced to belong to \(\Omega\) by a suitable control.

This definition is analogous to that of \((A, B)\)-invariant \([24]\) or controlled invariant \([1]\) subspaces, but in a more general framework.

It should be noticed that an \((A, B)\)-invariant set is a set which can be made positively invariant by an appropriate control action.

The following concept will be fundamental for the characterization of \((A, B)\)-invariance:

**Definition 3.3** The one-step admissible set of the set \(\Omega\) is defined as follows \([7]\):

\[
\mathcal{Q}(\Omega) = \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m : Ax + Bu \in \Omega\}.
\]

One can see that \(\mathcal{Q}(\Omega)\) is the set of all states which can be transferred to \(\Omega\) in one step. It is then clear that the \((A, B)\)-invariance of \(\Omega\) is equivalent to the following geometric condition:

**Theorem 3.1** \([17, 3]\) The set \(\Omega \subset \mathbb{R}^n\) is \((A, B)\)-invariant with respect to system (1) if and only if

\[
\Omega \subset \mathcal{Q}(\Omega).
\]

3.1 \((A, B)\)-invariance of polyhedra - general case

From now on, the study of the \((A, B)\)-invariance property will be limited to convex polyhedra containing the origin, that is to the case:

\[
\Omega = B[G, \rho] = \{x : Gx \leq \rho\}, \quad \rho \geq 0.
\]

For a given time \(k\), admissibility of the state vector at the time \(k + 1\) is characterized by the set of constraints:

\[
GAx(k) + GBu(k) \leq \rho. \tag{2}
\]

These constraints define a convex polyhedron, say \(\Pi\), on the linear space defined by the extended vector \(\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}\). The largest set of one-step admissible state vectors associated to (2) is then the projection of \(\Pi\) onto the state space. An explicit expression of this projection can be obtained from lemma 2.2:
\textbf{Proposition 3.1} The one-step admissible set, $Q(R[G, \rho])$, is the convex polyhedron $R[T GA, T \rho]$, where the row vectors of matrix $T$ form a minimal generating set of the nonnegative left kernel of matrix $GB$, defined by:

$$
\Gamma = \{ w \in \mathbb{R}^g : \ w \geq 0, \ (GB)^T w = 0 \}.
$$

\textbf{Proof:} Immediate from lemma 2.2. \( \square \)

From theorem 3.1 and proposition 3.1, $(A, B)$-invariance of $R[G, \rho]$ can then be geometrically characterized by:

$$
R[G, \rho] \subseteq R[T GA, T \rho].
$$

This characterization can be translated into matrix relations by means of the following result:

\textbf{Theorem 3.2} The convex polyhedron $R[G, \rho] \subseteq \mathbb{R}^n$ is $(A, B)$-invariant if and only if there exists a nonnegative matrix $Y$ such that:

$${\begin{array}{l}
YG = TGA, \\
Y\rho \leq T\rho.
\end{array}}$$

\textbf{Proof:} Immediate from condition (4) and lemma 2.3. \( \square \)

At this point, it is important to point out the advantages of the above characterization compared to those found in the literature. The major advantage is the fact that theorem 3.2 applies to any convex closed polyhedron, contrarily to the characterization proposed in [18, 7], which applies only to compact polyhedra.

The second advantage is of numerical nature. In the approach proposed in [18, 7], for testing for $(A, B)$-invariance, one needs initially to compute the vertices of the polyhedron, which is known to be a very hard computational task, mainly for large dimensional systems. Then, one has to test, at each vertex, for the existence of an admissible control. On the contrary, concerning theorem 3.2, once one has computed matrix $T$, computation for which efficient (and in general less expensive than those proposed for computation of vertices) methods are available, conditions (5), (6) can be checked by means of the resolution of a simple linear program.

The two following cases are important to point out:

- In the case $T = 0$, the polyhedron $R[T GA, T \rho]$ becomes the whole state space $\mathbb{R}^n$, and condition (4) is trivially satisfied. Relations (5), (6) are satisfied with $Y = 0$. The convex polyhedron $R[G, \rho]$ is then trivially $(A, B)$-invariant.

- In the case of an autonomous system ($B = 0$), $\Gamma$ is the entire nonnegative orthant $\mathbb{R}^g_+$, $T = I_g$, and relations (5), (6) become the classical positive invariance relations (see e.g. [4, 2, 20]).

Finally, it is interesting to note from relations (5), (6), that $(A, B)$-invariance of the characteristic cone $R[G, 0]$ is a necessary condition for $(A, B)$-invariance of $R[G, \rho]$. 

3.2 \( (A, B) \)-invariance of symmetrical polyhedra

The case of 0-symmetrical polyhedra, \( \Omega = S(Q, \mu) = \{ x : |Qx| \leq \mu \} \) is now considered. Note that \( S(Q, \mu) \) can be written in the form \( R[G, \rho] \) with \( G = \begin{bmatrix} Q & -Q \end{bmatrix} \), \( \rho = \begin{bmatrix} \mu \\ \mu \end{bmatrix} \).

Let \( [T_1 \ T_2] \) be a matrix whose rows form a minimal set of generators of the polyhedral cone \( \Gamma \) (3) (with \( G = \begin{bmatrix} Q & -Q \end{bmatrix} \)). Now, form the matrix \( T \) by deleting from matrix \( T_1 - T_2 \) the rows \( T_{1i} - T_{2i} \) for which \( T_{1i} - T_{2i} = 0 \) or \( T_{1i} - T_{2i} = -T_{1j} + T_{2j} \) for some \( j < i \). The following result specializes the \((A, B)\)-invariance relations to the symmetrical case:

**Corollary 3.1** The symmetrical convex polyhedron \( S(Q, \mu) \subset \mathbb{R}^n \) is \((A, B)\)-invariant with respect to system (1) if and only if there exists a matrix \( Y \) such that:

\[
YQ = TQA, \tag{7}
\]

\[
|Y| \mu \leq |T| \mu. \tag{8}
\]

**Proof:** From theorem 3.2, \( S(Q, \mu) \) is \((A, B)\)-invariant if and only if there exist nonnegative matrices \( Y_1 \) and \( Y_2 \) such that:

\[
(Y_1 - Y_2)Q = (T_1 - T_2)QA, \tag{9}
\]

\[
(Y_1 + Y_2)\mu \leq (T_1 + T_2)\mu. \tag{10}
\]

The rows \( i \) for which \( T_{1i} - T_{2i} = 0 \) need not be considered because in this case relations (9), (10) are trivially satisfied with \( Y_{1i} = Y_{2i} = 0 \). The same applies to the rows \( i \) for which \( T_{1i} - T_{2i} = -T_{1j} + T_{2j} \) for \( j < i \), because if for the row \( j \) there exist row vectors \( Y_{1j} \) and \( Y_{2j} \) verifying (9), (10), then the same relations will be verified for the row \( i \) with \( Y_{1i} = Y_{2j} \) and \( Y_{2i} = Y_{1j} \).

Consider now the matrix \( T \) and its rows \( T_i \). The following statements are true (otherwise the generators associated to the row \( i \) of \( T \) in \( [T_1 \ T_2] \) would not belong to the minimal generating set):

- \( T_{ij} = 0 \) only if the associated elements in \( T_1 \) and \( T_2 \) are null as well.
- \( T_{ij} > 0 \) (\(< 0\)) only if the associated elements in \( T_1 \) and \( T_2 \) are respectively positive (null) and null (positive).

In view of these facts, from relations (9), (10), \((A, B)\)-invariance of \( S(Q, \mu) \) is equivalent to the existence of nonnegative matrices \( Y_1 \) and \( Y_2 \) verifying:

\[
(Y_1 - Y_2)Q = TQA, \tag{11}
\]

\[
(Y_1 + Y_2)\mu \leq |T| \mu. \tag{12}
\]

Now, let \( Y = Y_1 - Y_2 \) and consider the matrices \( Y^+ \) and \( Y^- \) defined by:

\[
Y^+_{ij} = \max\{Y_{ij}, 0\}, \tag{13}
\]

\[
Y^-_{ij} = \max\{-Y_{ij}, 0\}. \tag{14}
\]
Necessity of (7), (8) follows from the fact that the above matrices are such that \( Y^+ - Y^- = Y = Y_1 - Y_2 \) and \((Y^+ + Y^-)\mu = [Y]_1 + [Y_2] \mu \leq \|T\| \mu \). Sufficiency follows from the fact that \( Y^+ \) and \( Y^- \) verify relations (11), (12).

Note that any row vector \( t \) belonging to the left kernel of map \( QB \) can be written in the form \( t = t_1 - t_2; \ t_1, t_2 \geq 0 \). This means that \([t_1 \ t_2]^T \) belongs to \( \Gamma \) \((3)\), with \( G = \begin{bmatrix} Q & -Q \end{bmatrix} \).

Therefore, matrix \( T \) necessarily contains as a sub-matrix a row-vector basis of the left kernel of map \( QB \).

In view of this fact, as a complementary result, corollary 3.1 can be specialized to the case of vector subspaces as follows:

**Corollary 3.2** The subspace \( \ker(Q) \in \mathbb{R}^n \) is \((A, B)\)-invariant if and only if there exists a matrix \( M \) such that: \( MQ = KQA \), where the row vectors of \( K \) span the left kernel of map \( QB \).

**Proof:** From corollary 3.1, \( \ker(Q) = S(Q, 0) \) is \((A, B)\)-invariant if and only if there exists a matrix \( Y \) verifying (7) \((\text{condition (8) is trivially verified})\). Since the row vectors of \( T \) span \( \ker(QB) \), then condition (7) can be rewritten in the form:

\[
\begin{bmatrix} M \\ N \end{bmatrix} Q = \begin{bmatrix} K \\ LK \end{bmatrix} QA,
\]

where the row vectors of \( K \) are selected from \( T \) so as to span the left kernel of \( QB \). Since \( MQ = KQA \), then \( NQ = LKQA \) is satisfied with \( N = LM \).

Consider now the factor space \( \bar{X} = \mathbb{R}^n_{\text{Im}(Q)} \) and let \( P : \mathbb{R}^n \to \bar{X} \) be the canonical projection. The following maps can be defined on \( \bar{X} \) \([24]\):

- \( \bar{A} \), the map induced in \( X \) by \( A \), given by \( \bar{A}P = PA \),
- \( \bar{B} \), given by \( \bar{B} = PB \),
- \( \bar{Q} \), given by \( \bar{Q}P = Q \).

Then, the following "factor system" can be defined in \( \bar{X} \) \([24]\):

\[
\bar{x}(k + 1) = \bar{A}\bar{x}(k) + \bar{B}u(k),
\]

as well as the polyhedron:

\[
\mathcal{S}(\bar{Q}, \mu) = \{ \bar{x} : |\bar{Q}\bar{x}| \leq \mu \},
\]

and its extension to \( \mathbb{R}^n \):

\[
S(\bar{Q}, \mu) = \mathcal{S}(\bar{Q}, \mu)P.
\]

The polyhedron \( S(Q, \mu) \) can then be decomposed in the following form \([22]\):

\[
S(Q, \mu) = \ker(Q) + S(\bar{Q}, \mu).
\]

The following result shows that the test for \((A, B)\)-invariance of unbounded symmetrical polyhedra can be decomposed.

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Corollary 3.3 The symmetrical polyhedron $S(Q, \mu)$ is $(A, B)$-invariant with respect to system (1) if and only if:

1. The subspace $\ker(Q)$ is $(A, B)$-invariant,
2. The compact polyhedron $\tilde{S}(\tilde{Q}, \mu)$ is $(\tilde{A}, \tilde{B})$-invariant with respect to system (15).

Proof: Suppose initially that $S(Q, \mu)$ is $(A, B)$-invariant. $(A, B)$-invariance of $\ker(Q)$ follows from (7), from the fact that the rows of $T$ span the left kernel of $QB$, and from corollary 3.2. Again from (7), there exists a matrix $Y$ such that $Y\tilde{P} = T\tilde{P}PA = T\tilde{Q}A$, hence, $Y\tilde{Q} = T\tilde{Q}A$. This relation, together with (8), shows the $(\tilde{A}, \tilde{B})$-invariance of $\tilde{S}(\tilde{Q}, \mu)$.

Suppose conversely that $\ker(Q)$ is $(A, B)$-invariant and that $\tilde{S}(\tilde{Q}, \mu)$ is $(\tilde{A}, \tilde{B})$-invariant with respect to system (15). From (18), every vector $x \in S(Q, \mu)$ can be written in the form $x = x^Q + x^\tilde{Q}$, with $x^Q \in \ker(Q)$, $x^\tilde{Q} \in S(\tilde{Q}, \mu)$. By assumption, there exists a matrix $F^Q$ such that $\ker(Q)$ is $(A + BF^Q)$-invariant, and a control sequence $\{u^\tilde{Q}(k)\}$, $k \in \mathcal{N}$ such that $x^\tilde{Q}(k) \in \ker(Q) + S(\tilde{Q}, \mu)$. Then, it is clear that $S(Q, \mu)$ is positively invariant under the control law $u(k) = F^Qx^Q(k) + u^\tilde{Q}(k)$. □

3.3 $(A, B)$-\(\lambda\)-contractive sets

In the case of compact $(A, B)$-invariant sets containing the origin, it is often important to increase the convergence rate of the trajectory to the equilibrium point. For example, this can help the system incorporate the effects of disturbances and/or uncertainties. $(A, B)$-invariance and convergence rate are conjugated in the following definition.

Definition 3.4 Given $0 < \lambda \leq 1$, a compact set $\Omega \subset \mathbb{R}^n$ is said to be $(A, B)$-invariant \(\lambda\)-contractive (or simply $(A, B)$-\(\lambda\)-contractive) with respect to system (1) if for all $x \in \Omega$ there exists a control vector $u \in \mathbb{R}^m$ such that $Ax + Bu \in \lambda \Omega$.

It is clear that an $(A, B)$-invariant set is an $(A, B)$-\(\lambda\)-contractive set with $\lambda = 1$.

In the case where the origin belongs to the interior of a convex polyhedron $R[G, \rho]$ ($\rho > 0$), the condition for one-step admissibility (2) can be replaced by the $(A, B)$-\(\lambda\)-contractivity condition: $GAx(k) + GBu(k) \leq \lambda \rho$, which results in the following:

- Given a contraction rate $\lambda$, the one-step admissible set of the polyhedron $R[G, \rho]$ is the convex polyhedron $R[TGA, \lambda T\rho]$, where matrix $T$ is defined as in theorem 3.2.
- The convex polyhedron $R[G, \rho] \subset \mathbb{R}^n$ is $(A, B)$-\(\lambda\)-contractive with respect to system (1) if and only if there exists a non-negative matrix $Y$ such that (5) is verified and:

$$Y\rho \leq \lambda T\rho,$$

$$\text{(19)}$$

- The convex symmetrical polyhedron $S(Q, \mu) \subset \mathbb{R}^n$ is $(A, B)$-\(\lambda\)-contractive with respect to system (1) if and only if there exists a non-negative matrix $Y$ such that (7) is verified and:

$$\|Y\mu\leq \lambda \mu, \text{(20)}$$
4 The Supremal \((A, B)\)-invariant Set

Suppose now that the state of system (1) is subject to the constraint \(x \in \Omega\). The set \(\Omega\) is not \((A, B)\)-invariant in general, hence there are states in \(\Omega\) for which this constraint cannot be satisfied in one-step. A possible solution to this problem is to restrict the state to an \((A, B)\)-invariant set contained in \(\Omega\). It is also desirable that this set be as large as possible. Such a set is said to be \textit{supremal}, and its existence is a consequence of the following property, whose proof is straightforward:

**Proposition 4.1** The family of all \((A, B)\)-invariant sets contained in a convex set \(\Omega\) is closed under the operation "convex hull of the union".

Since \(\Omega\) is closed by assumption, this proposition guarantees the existence, in the family of \((A, B)\)-invariant sets contained in \(\Omega\), of a supremal element (an element which contains all the other elements):

\[
C^\infty(\Omega) \triangleq \text{supremal } (A, B)\text{-invariant set contained in } \Omega.
\]

Indeed, \(C^\infty(\Omega)\) is the set defined by the convex hull of the union of all \((A, B)\)-invariant sets in \(\Omega\). The supremal set can be characterized by the following recurrence formula [7]:

\[
C^{i+1} = \mathcal{Q}(C^i) \cap C^i, \quad \text{with } C^0 = \Omega, \quad (21)
\]

\[
C^\infty(\Omega) = \lim_{i \to \infty} C^i. \quad (22)
\]

It should be noticed that the set \(C^i\) is the set of states for which there exists a control sequence able to force them to stay in \(\Omega\) in \(i\) steps. According to (21), \(C^i \subseteq C^{i-1}\), and the supremal set is obtained for \(i \to \infty\).

One can also introduce a contraction rate \(\lambda\), by computing the set:

\[
C^\infty(\Omega, \lambda) \triangleq \text{supremal } (A, B)\text{-contractive set contained in } \Omega.
\]

For doing so, it suffices to replace the recurrence (21), (22) by:

\[
C^{i+1} = \mathcal{Q}(\lambda C^i) \cap C^i, \quad \text{with } C^0 = \Omega, \quad (23)
\]

\[
C^\infty(\Omega, \lambda) = \lim_{i \to \infty} C^i. \quad (24)
\]

Let us now concentrate on the computation of \(C^\infty(\Omega)\) for \(\Omega = R[G, \rho]\).

In [19], a recurrence formula different from (21), (22) is used. The associated numerical method is however very complex, for it uses alternately the representations of polyhedra in terms of vertices and in terms of linear inequalities. Furthermore, matrix \(A\) must be invertible. In [7], formula (21), (22) is used, but under the assumption that \(R[G, \rho]\) is compact (since \((A, B)\)-invariance is tested vertex by vertex).
From proposition 3.1, the set \( Q(R[G, \rho]) \), coincides with the polyhedron \( R[TGA, T\rho] \). This results allow for the construction of the supremal \((A, B)\)-invariant set contained in \( R[G, \rho] \) by means of recurrence (21), (22). Such a construction demands in general a very large computational effort. Along the iterative process, many redundant inequalities can be generated. It is therefore particularly desirable to implement an algorithm which only generates non-redundant inequalities (with respect to the set of inequalities computed in the preceding iteration) at each iteration. This feature can be achieved by using the \((A, B)\)-invariance relations (5), (6), as shown through the following algorithm [11]:

1. Initialize \( i = 1, \; P^0 = 0, \; T^0 = 0, \; \rho^0 = 0, \; G^0 = G, \; \rho^0 = \rho, \; g^0 = g \). Define a precision \( \epsilon \).

2. Compute the matrix \( T^i \in \mathbb{R}^{i \times s^i} \), whose rows form a generating set of the polyhedral cone \( \Gamma^i = \{ w; \; (G^i B)^T w = 0, \; w \geq 0 \} \) and decompose \( T^i \) in the form:

\[
T^i = \begin{bmatrix} T_{i-1} & 0 \\ U^i & \end{bmatrix}; \quad U^i \in \mathbb{R}^{i \times s^i}.
\]  

(25)

3. Solve by linear programming the following problems, for \( j = 1, \ldots, r^i \):

\[
\begin{align*}
\min_{Y^i_j} & \; Y^i_j \rho^i_j \\
\text{subject to:} & \; Y^i_j G^i = U^i_j G^i A, \quad Y^i_j \geq 0 \quad \text{. (26)}
\end{align*}
\]

If \( \forall j \), \( Y^i_j \rho^i - U^i_j \rho^i \leq \epsilon \), then \( C^\infty (R[G, \rho]) = R[G^i, \rho^i] \), STOP.

If \( \exists j : Y^i_j \rho^i - U^i_j \rho^i > \epsilon \), order the rows of \( U^i \) and \( Y^i \), by means of a permutation matrix \( P^i \), in the form:

\[
P^i U^i = \begin{bmatrix} U^i_1 \\ U^i_2 \end{bmatrix}, \quad P^i Y^i = \begin{bmatrix} Y^i_1 \\ Y^i_2 \end{bmatrix},
\]

with \( Y^i_2 \in \mathbb{R}^{\ell \times r^i} \), so that:

\[
\begin{align*}
Y^i_1 \rho^i - U^i_1 \rho^i & \leq \epsilon 1, \\
Y^i_2 \rho^i - U^i_2 \rho^i & > \epsilon 1. \quad \text{ (27), (28)}
\end{align*}
\]

4. Construct the matrices

\[
G^{i+1} = \begin{bmatrix} G^i \\ U^i_2 G^i A \end{bmatrix}, \quad \rho^{i+1} = \begin{bmatrix} \rho^i \\ U^i_2 \rho^i \end{bmatrix}.
\]

Do \( g^{i+1} = g^i + \tilde{\rho} \).

5. Do \( i = i + 1 \) and return to step 2.

\[\Box\]

The fact that this algorithm converges to \( C^\infty (R[G, \rho]) \) follows from the next result:
Proposition 4.2. The polyhedron \( R[G^i, \rho^i] \) is identical to the set \( \mathcal{Q}(R[G^i, \rho^i]) \cap R[G^i, \rho^i] \).

Proof: From proposition 3.1 and from (25), the set \( \mathcal{Q}(R[G^i, \rho^i]) \cap R[G^i, \rho^i] \) is defined by the inequalities:

\[
\begin{align*}
G^i x & \leq \rho^i \\
U_1^i G^i Ax & \leq U_1^i \rho^i \\
U_2^i G^i Ax & \leq U_2^i \rho^i
\end{align*}
\]

Note that the rows associated to \( T^i \) in (25) have already been considered in the computation of \( R[G^i, \rho^i] \).

The inclusion of \( \mathcal{Q}(R[G^i, \rho^i]) \cap R[G^i, \rho^i] \) in \( R[G^i+1, \rho^i+1] \) is evident. Conversely, every point \( x \in R[G^i+1, \rho^i+1] \) verifies, with the non-negative matrix \( Y_1^i \) verifying (26), (27): \( U_1^i G^i Ax = Y_1^i G^i x \leq Y_1^i \rho^i \leq U_1^i \rho^i + \epsilon 1 \), which implies, up to the given precision: \( R[G^i+1, \rho^i+1] \subset \mathcal{Q}(R[G^i, \rho^i]) \cap R[G^i, \rho^i] \). □

Concerning this algorithm, the following should be pointed out:

- For some cases (namely when \( \text{rank}(G^i) < n \)) the LP (26) may not be solvable for some values of \( j \). The associated rows \( U_2^i \) and \( Y_2^i \) must in this case be included in matrices \( U_2^i \) and \( Y_2^i \) (28) respectively.

- The set \( C^\infty(\Omega) \) is polyhedral if and only if it is generated in a finite number of iterations. It can however be approximated by a polyhedron. Indeed, it has been shown in [7] that, given the set \( C^\infty(\Omega, \lambda) \), \( 0 < \lambda < 1 \), then \( \forall \lambda' : \lambda \leq \lambda' \leq 1, \exists \mathcal{C}^i \) is \( \lambda' \)-contractive, with \( \mathcal{C}^i \) given by the recurrence (23). An approximation of \( C^\infty(\Omega) \) can therefore be computed through the recurrence (23), with \( \lambda \) close to 1, until eventually obtaining an \((A, B)\)-invariant polyhedron \( C^i \).

It is interesting to notice that for unbounded symmetrical polyhedra, the computation of \( C^\infty(\Omega) \) can be performed in a decomposed manner. Let then \( S(Q, \mu) \) be an unbounded symmetrical polyhedron and let \( V^* \) be the largest \((A, B)\)-invariant subspace contained in \( \ker(Q) \). Consider also the factor space \( \tilde{X} = \frac{\mathbb{R}^n}{\mathbb{Z}^n} \) and the reduced order system:

\[
\dot{\tilde{x}}(k+1) = \tilde{A} \tilde{x}(k) + \tilde{B}u(k).
\]

The maps \( \tilde{A} \) and \( \tilde{B} \) are defined from the orthogonal projection \( P_v : \mathbb{R}^n \to \tilde{X} \), the same way as their analogs in system (15). The polyhedron \( S(Q, \mu) \) is now decomposed in the following form:

\[
S(Q, \mu) = V^* + S(\hat{Q}, \mu),
\]

where \( S(\hat{Q}, \mu) = \hat{S}(\hat{Q}, \mu) P_v \), with \( \hat{S}(\hat{Q}, \mu) \Delta \{ \tilde{x} ; |\tilde{Q}\tilde{x}| \leq \mu \} \).

Define also the set

\[
\hat{C}^\infty(\hat{S}(\hat{Q}, \mu)) \triangleq \text{supremal} (\hat{A}, \hat{B})\text{-invariant set contained in } \hat{S}(\hat{Q}, \mu),
\]

and its extension to \( \mathbb{R}^n \),

\[
\hat{C}^\infty(S(Q, \mu)) \Delta \hat{C}^\infty(\hat{S}(\hat{Q}, \mu)) P_v.
\]
Theorem 4.1

\[ \mathcal{C}^\infty (S(Q, \mu)) = \mathcal{V}^* + \hat{\mathcal{C}}^\infty (S(Q, \mu)). \]

**Proof:** \( \forall x \in \mathcal{V}^* + \hat{\mathcal{C}}^\infty (S(Q, \mu)), \) there exist \( x^v \in \mathcal{V}^* \) and \( x^c \in \hat{\mathcal{C}}^\infty (S(Q, \mu)) \) such that \( x = x^v + x^c. \) From the construction of \( \mathcal{V}^* \) and \( \hat{\mathcal{C}}^\infty (S(Q, \mu)), \) \( \exists F^v \) such that \( \mathcal{V}^* \) is \( (A + BF^v) \)-invariant and there exists a control sequence \( \{u(k)\}, k \in \mathcal{N} \) such that \( x^c(k) \in \mathcal{V}^* + \hat{\mathcal{C}}^\infty (S(Q, \mu)). \) Hence, \( \mathcal{V}^* + \hat{\mathcal{C}}^\infty (S(Q, \mu)) \) is an \( (A, B) \)-invariant set contained in \( S(Q, \mu). \)

Conversely, consider a generic point \( x(k) \in \mathcal{C}^\infty (S(Q, \mu)). \) Since \( \mathcal{C}^\infty (S(Q, \mu)) \) is \( (A, B) \)-invariant, then there exists a control \( u(k) \) such that \( x(k+1) \in S(Q, \mu). \) The vector \( x(k) \in S(Q, \mu) \) can be decomposed in the form: \( x(k) = x^v(k) + x^s(k) \) with \( x^v(k) \in \mathcal{V}^* \) and \( x^s(k) \in S(Q, \mu). \) The control vector \( u(k) \) can be decomposed in the form \( u(k) = F^v x^v(k) + u^s(k), \) with \( \mathcal{V}^* \) \( (A + BF^v) \)-invariant. Therefore, \( x^v(k+1) = (A + BF^v)x^v(k) \in \mathcal{V}^* \). Consider now the system (29) and suppose \( x^s(k) \not\in \hat{\mathcal{C}}^\infty (S(Q, \mu)). \) One can the verify that \( \hat{x}^s(k+1) = \hat{A}\hat{x}^s(k) + \hat{B}u(k) \not\in \hat{S}(Q, \mu), \) which contradicts the assumption \( x(k) \in \mathcal{C}^\infty (S(Q, \mu)). \) Hence, \( \mathcal{C}^\infty (S(Q, \mu)) \subset \mathcal{V}^* + \hat{\mathcal{C}}^\infty (S(Q, \mu)) \) and the proof is completed. \( \square \)

## 5 Computation of a Control Law

The satisfaction of the \( (A, B) \)-invariance relations presented beforehand guarantees, for all point in the considered polyhedron, the existence of a control which forces the state trajectory to stay in it. That does not presuppose however a particular type of control law. It is nevertheless important, even mandatory in practice, that the system be controlled by means of a closed-loop control law.

An answer to this question has been given in [18] and improved in [7]: a piece-wise linear control law was proposed to achieve closed-loop positive invariance of compact \( (A, B) \)-invariant polyhedra. To this end, the polyhedral sets are divided in regions defined by the convex hull of the origin and \( n \) vertices (where \( n \) is the dimension of the system). A state feedback control law is then computed for each region.

The control law described in the sequel extends such a law to the case of non-compact polyhedra.

### 5.1 General case

Every polyhedron \( R[G, \rho] \) can be decomposed in the form of a sum of the characteristic cone \( R[G, 0] \) and a polytope \( \Pi \) \([22] \). A set of admissible controls \( (w_1, \ldots, w_p) \) can then be associated to the vertices \( (x_1, \ldots, x_p) \) of \( \Pi \). They verify: \( G Ax_i + GB w_i \leq \rho. \) Similarly, a set of controls \( (w_1, \ldots, w_q) \) can be associated to the set of generators \( M_j \) of \( R[G, 0] \), so that: \( GAM_j + GB w_j \leq \rho. \)

Each point in \( R[G, \rho] \) is represented by the set of coordinates \( (\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p) \), through the relation:

\[ x = \sum_{j=1}^{q} \alpha_j M_j + \sum_{i=1}^{p} \beta_i x_i, \]  
with \( \alpha_j \geq 0 \) \( \forall j \), \( 0 \leq \beta_i \leq 1 \) \( \forall i \), \( \sum_{i=1}^{p} \beta_i \leq 1. \]  

(30)
The following control function can then be considered:

\[
u(x) = \sum_{j=1}^{q} \alpha_j w_j + \sum_{i=1}^{p} \beta_i v_i.\tag{31}\]

For \(\rho \geq 0\), a partition of \(R[G, \rho]\) can be derived from the parameterization (30). Each region \(X_r\) of \(R[G, \rho]\) is generated from relation (30) by a set of generators and/or vertices \((M_{j'}, x_{i'}), j' \in J_r, i' \in I_r\), such that:

- \(\text{card}(J_r) + \text{card}(I_r) = n\), where \(\text{card}(\cdot)\) represents the number of elements in the considered set.
- A point \(x \in X_r\) is given by:

\[
x = \sum_{j' \in J_r} \alpha_{j'} M_{j'} + \sum_{i' \in I_r} \beta_{i'} x_{i'}, \text{ with } \alpha_{j'} \geq 0, 0 \leq \beta_{i'} \leq 1, \sum_{i' \in I_r} \beta_{i'} \leq 1.\tag{32}\]

The transition between two adjacent regions is characterized by a pivoting operation for which one of the coefficients \((\alpha_{j'}, \beta_{i'})\) vanishes and, either a generator \(M_{j'}, j' \notin J_r\), or a vertex \(x_{i'}, i' \notin I_r\) replaces in the representation (32) the generator or vertex for which either \(\alpha_{j'}\) or \(\beta_{i'}\) has vanished. The interior of the intersection of two adjacent regions is empty, and the union of all regions is the polyhedron \(R[G, \rho]\).

Let now \(X_r\) be a square matrix whose columns are the generators/vertices which define the region \(X_r\), and let \(U_r\) be a matrix whose columns are the associated control vectors \(w_j, v_i\). A piecewise linear control law is then given by:

\[
u(k) = F_r x(k) = U_r (X_r)^{-1} x(k), \text{ for } x(k) \in X_r.\tag{33}\]

This law is a possible realization of the law (31).

Since compact polyhedra are completely defined by their vertices, the regions in which they are divided are compact polyhedra formed by the convex hull of the origin and \(n\) vertices. In this case, the control law (33) becomes that proposed in [18, 7].

5.2 Symmetrical polyhedra

For unbounded symmetrical polyhedra, the control law can be computed in a decomposed manner. Consider then the unbounded symmetrical polyhedron \(S(Q, \mu)\) and matrix \(P\) defined, as in section 3.2, as the canonical projection of \(\mathbb{R}^n\) onto the factor space \(\tilde{X} = \mathbb{R}^n / \ker(Q)\).

The following result can be established:

**Corollary 5.1** If the polyhedron \(S(Q, \mu)\) is \((A, B)\)-invariant with respect to system (1), then a control law such that it is positively invariant in closed-loop is given by:

\[
u(k) = F^Q x(k) + u^S(k),\]

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where \( P^Q \) is such that \( \ker(Q) \) is \((A + BF^Q)\)-invariant, and \( u^S(k) \) is a control law such that the compact polyhedron \( \bar{S}(\bar{Q}, \mu) \) is positively invariant with respect to the system
\[
\bar{x}(k+1) = (\bar{A} + \bar{B}F^Q)\bar{x}(k) + \bar{B}u(k),
\]
where \( F^Q \) is defined by: \( \tilde{F}^Q P = F^Q \).

**Proof:** Suppose that \( x(k) \in S(Q, \mu) \). Under the proposed control law, one has
\[
x(k+1) = (A + BF^Q)x(k) + Bu^S(k).
\]
Hence \( P\bar{x}(k+1) = \bar{x}(k+1) = P(A + BF^Q)x(k) + PBu^S(k) = (\bar{A} + \bar{B}F^Q)\bar{x}(k) + \bar{B}u^S(k) \).

Then, by assumption, \( u^S(k) \) is such that \( \bar{x}(k+1) \in \bar{S}(\bar{Q}, \mu) \). The existence of \( u^S(k) \) follows from corollary 3.3 and from the fact that if \( \bar{S}(\bar{Q}, \mu) \) is \((\bar{A}, \bar{B})\)-invariant, then it is \((\bar{A} + \bar{B}F^Q, \bar{B})\)-invariant as well. Now, since \( \bar{x}(k+1) \in \bar{S}(\bar{Q}, \mu) \), then \( x(k+1) \in \ker(Q) + S(\bar{Q}, \mu) = S(Q, \mu) \).

\[\square\]

6 Extensions

The preceding results can be readily extended to two important classes of systems: systems subject to control constraints and persistently disturbed systems.

6.1 Systems subject to control constraints

Consider again system (1) whose control vector is now subject to the following constraints:

\[
u(k) \in \mathcal{U} \subset \mathbb{R}^m \; \forall k \in \mathcal{N}
\]

**Definition 6.1** A non-empty closed set \( \Omega \subset \mathbb{R}^m \) is said to be \( \mathcal{U}-(A, B)\)-invariant with respect to system (1), (34) if \( \forall x \in S \), there exists a control vector \( u \in \mathcal{U} \) such that \( Ax + Bu \in \Omega \).

The one-step admissible set is now defined as [7]:
\[
\mathcal{Q}(\Omega, \mathcal{U}) = \{ x \in \mathbb{R}^n : \exists u \in \mathcal{U} : Ax + Bu \in \Omega \}.
\]

Therefore, the set \( \Omega \) is \( \mathcal{U}-(A, B)\)-invariant with respect to (1), (34) if and only if \( \Omega \subset \mathcal{Q}(\Omega, \mathcal{U}) \).

Let us now study the \( \mathcal{U}-(A, B)\)-invariance of polyhedral sets \( R[G, \rho] \) with respect to systems subject to linear control constraints (34) with:
\[
\mathcal{U} = R[U, \psi] = \{ u \in \mathbb{R}^m : Uu \leq \psi \}.
\]

For a given \( k \), admissibility of the state vector in \( k + 1 \) is characterized by the following inequalities:
\[
GAx_k + GBu_k \leq \rho,
\]
\[
Uu_k \leq \psi.
\]

The one-step admissible set can then be characterized as follows:
Proposition 6.1

\[ Q(R[G, \rho], R[U, \psi]) = R[T_3GA, T_3\rho + T_u\psi], \]

where the rows of matrix \([T_3 \ U] \) form a minimal generating set of the nonnegative left kernel of matrix \( \begin{bmatrix} GB \\ U \end{bmatrix} \), defined by:

\[ \Gamma_u = \left\{ \begin{bmatrix} w_3 \\ w_u \end{bmatrix} : \begin{bmatrix} w_3 \\ w_u \end{bmatrix} \geq 0, \ [(GB)^T \ UT] \begin{bmatrix} w_3 \\ w_u \end{bmatrix} = 0, \right\}. \] (37)

Proof: Immediate from conditions (36) and from lemma 2.2. □

The \( U-(A, B)\)-invariance of \( R[G, \rho] \), with \( U = R[U, \psi] \), can therefore be characterized by the following geometric condition:

\[ R[G, \rho] \subset R[T_3GA, T_3\rho + T_u\psi], \] (38)

and by the algebraic condition given by the following proposition:

Theorem 6.1 The convex polyhedron \( R[G, \rho] \) is \( U-(A, B)\)-invariant with respect to system (1), (34), with \( U = R[U, \psi] \), if and only if there exists a non-negative matrix \( Y \) such that:

\[ YG = T_3GA, \] (39)

\[ Y \rho \leq T_3\rho + T_u\psi. \] (40)

Proof: Immediate from condition (38) and from lemma 2.3. □

From this result, one can notice that \((A, B)\)-invariance of \( R[G, \rho] \) is a necessary condition for its \( U-(A, B)\)-invariance. This follows from the fact that the rows of matrix \([T \ 0] \), where the rows of \( T \) form a generating set of the polyhedral cone \( \Gamma \) (3), belong to the cone \( \Gamma_u \) (37).

As for the unconstrained case, it can be shown the existence of a supremal \( U-(A, B)\)-invariant set contained in a given set \( \Omega \):

\[ C^\infty(\Omega, U) \overset{\Delta}{=} \text{supremal } U-(A, B)\text{-invariant set contained in } \Omega. \]

From the definition of the one-step admissible set, and similarly to the unconstrained case, it can be seen that this supremal set is given by the following recurrence:

\[ C_{i+1} = Q(C_i, U) \cap C_i, \text{ with } C_0 = \Omega, \]

\[ C^\infty(\Omega, U) = \lim_{i \to \infty} C_i. \]

An algorithm for computing \( C^\infty(\Omega, U) \) with \( \Omega = R[G, \rho], U = R[U, \psi] \) can therefore be easily derived from algorithm of section 4.
### 6.2 Persistently disturbed systems

Consider the following linear discrete-time system:

\[ x(k + 1) = Ax(k) + Bu(k) + Ed(k), \]  

where \( d \in \mathbb{R}^q \) is a disturbance vector, supposed constrained to evolve inside a bounded domain \( \mathcal{D} \subset \mathbb{R}^q \):

\[ d(k) \in \mathcal{D} \quad \forall k \in \mathcal{N}. \]

One can notice that this kind of disturbance acts continuously in time, and its energy is infinite. This is why it is named by some authors **persistent disturbances**.

**Definition 6.2** A non-empty closed set \( \Omega \subset \mathbb{R}^n \) is said to be \( \mathcal{D}-(A,B)\)-invariant with respect to system (41), (42) if \( \forall x \in \Omega \) there exists a control vector \( u \), such that \( Ax + Bu + Ed \in \Omega \quad \forall d \in \mathcal{D} \).

This definition assumes that the disturbance vector is not measured. The case of measurable disturbances has been considered in [12].

The one-step admissible set is now defined as [7]:

\[ Q(\Omega, \mathcal{D}) = \{ x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m : Ax + Bu + Ed \in \Omega, \ \forall d \in \mathcal{D} \}. \]

Therefore, the set \( \Omega \) is \( \mathcal{D}-(A,B)\)-invariant with respect to (41), (42) if and only if \( \Omega \subset Q(\Omega, \mathcal{D}) \).

Consider now the polyhedral case: \( \Omega = R[G, \rho] , \mathcal{D} = R[D, \omega] = \{ d \in \mathbb{R}^q : Dd \leq \omega \} \). Define the components \( \delta_i \) of vector \( \delta \) as follows:

\[ \delta_i = \max_{d \in R[D, \omega]} G_i Ed. \]

For a given \( k \), admissibility of the state vector in \( k + 1 \) is now characterized by:

\[ GAx(k) + GBu(k) \leq \rho - \delta \]

One can easily notice that the role of vector \( \delta \) is to absorb the effect of the disturbances. The following results can then be easily derived:

**Proposition 6.2**

\[ Q(R[G, \rho], R[D, \omega]) = R[TA, T(\rho - \delta)], \]

where the rows of matrix \( T \) form a minimal generating set of the nonnegative left kernel of matrix \( GB \).

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Theorem 6.2  The convex polyhedron \( R[G, \rho] \) is \( D-(A, B) \)-invariant with respect to system \((41), (42)\), with \( D = R[D, \omega] \), if and only if there exists a nonnegative matrix \( Y \) such that:

\[
YG = TGA, \quad (45)
\]

\[
Y \rho \leq T(\rho - \delta). \quad (46)
\]

As for the undisturbed case, it can be shown the existence of a supremal \( D-(A, B) \)-invariant set contained in a given set \( \Omega \):

\[
C^\infty(\Omega, \mathcal{U}) \triangleq \text{supremal } \mathcal{U}-(A, B) \text{-invariant set contained in } \Omega,
\]

which is given by:

\[
C_{i+1} = Q(C_i, \mathcal{D}) \cap C_i, \quad \text{with } C_0 = \Omega,
\]

\[
C^\infty(\Omega, \mathcal{D}) = \lim_{i \to \infty} C_i.
\]

An algorithm for computing \( C^\infty(\Omega, \mathcal{D}) \) with \( \Omega = R[G, \rho], \mathcal{U} = R[U, \psi] \) can also be easily derived from algorithm of section 4.

A practical application of \( D-(A, B) \)-invariant polyhedra is in the solution of the persistent disturbance attenuation problem, also known as the \( \ell^1 \) control problem (see e.g. [8, 14, 16, 15]).

7  Numerical Example

Consider the system \((1)\) for which:

\[
A = \begin{bmatrix} 0.4 & 0.9 \\ 0.6 & 1.8 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

and the polyhedron \( R[G, \rho] \) with

\[
G = \begin{bmatrix} 0.2 & 0.2 \\ -1 & -1 \\ -1 & 0.35 \\ 0.25 & -0.5 \end{bmatrix}, \quad \rho = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]

The computation of the largest \( (A, B) \)-\( \lambda \)-contractive set contained in \( R[G, \rho] \), with \( \lambda = 0.8 \), results in \( C^\infty(R[G, \rho], \lambda) = R[G^1, \rho^1] \), with:

\[
G^1 = \begin{bmatrix} G \\ 0.6 & 1.35 \\ -1.5429 & -3.4714 \end{bmatrix}, \quad \rho^1 = \begin{bmatrix} \rho \\ 5.6 \\ 3.0857 \end{bmatrix}.
\]
For $R[G^1,\rho^1]$, a matrix $T^1$ whose rows generate the non-negative left kernel of $G^1 B$, and a matrix $Y$ which verifies the conditions of $(A, B)$-λ-contractivity (5), (19) are given by:

$$
T^1 = \begin{bmatrix}
5 & 1 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 2 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0.2881 \\
0 & 1 & 2.8571 & 0 & 0 & 0 \\
0 & 0 & 2.8571 & 2 & 0 & 0 \\
0 & 0 & 2.8571 & 0 & 0 & 0.2881 \\
0 & 1 & 0 & 0 & 0.7407 & 0 \\
0 & 0 & 0 & 2 & 0.7407 & 0 \\
0 & 0 & 0 & 0 & 0.7407 & 0.2881 \\
\end{bmatrix}, \quad Y^1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.3889 \\
0 & 0 & 0 & 0 & 0 & -0.1440 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0.6111 \\
0 & 0 & 0 & 0 & 0 & 0.8560 \\
0 & 0 & 0 & 0 & 0 & 0.1440 \\
0 & 0 & 0 & 0 & 0 & -0.2449 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
$$

The polyhedral sets $R[G,\rho]$ and $R[G^1,\rho^1]$ are represented in figure 1. In figure 2 is represented the set $R[G^1,\rho^1]$ divided in 6 regions, as described in section 5.1. A trajectory of the state starting from one of the vertices, obtained through the application of a control law of the same type as (33) is represented as well.

**FIG. 1 - $R[G,\rho]$ and $C^\infty (R[G,\rho],\lambda)$.**

**FIG. 2 - $C^\infty (R[G,\rho],\lambda)$ divided in regions.**

### 8 Conclusion

This work has studied the concept of $(A, B)$-invariance applied to polyhedral sets of the state space of linear systems. This concept has proven to be of fundamental importance to the control of constrained systems. An explicit characterization of $(A, B)$-invariance for discrete-time systems has been proposed, which amounts to necessary and sufficient conditions in the form of linear matrix relations. The advantages of such a characterization, when compared to the ones found in the literature, are twofold: It applies to any convex polyhedron and it does not demand the computation of vertices. These advantages are particularly felt in computing the supremal $(A, B)$-invariant set contained in a given polyhedron. A numerical method has also been proposed to this computation, which uses the $(A, B)$-invariance relations to generate only non-redundant inequalities and to furnish an efficient test for convergence.

All these results have been specialized to the case of 0-symmetrical polyhedral sets. In particular, it has been shown that $(A, B)$-invariance of an unbounded symmetrical polyhedron
is equivalent to the \((A, B)\)-invariance of a subspace plus the \((A, B)\)-invariance of a compact polyhedron associated to a reduced order system.

The problem of computing a control law which achieves closed-loop positive invariance of an \((A, B)\)-invariant polyhedron has also been considered. A piecewise linear state feedback control law, proposed in the literature for compact polyhedra, has then been proposed to the general case. The implementation of this law is very complex though. We think that the search for simpler control laws should continue.

It has also been shown how the \((A, B)\)-invariance results can be extended to two cases which are potentially important in practice: systems subject to control constraints and to bounded additive disturbances.

An extension of this work to continuous-time systems can be found in [13].

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References


