Conditions of Stability via Positive Invariance for Delay Systems

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ABSTRACT. Considered delay systems are represented by a linear delay differential equation or by a linear delay difference equation. The system parameters and the delays are assumed unperfectly known. The instantaneous state vector (in the first model case) or the output vector (in the second model case) is perturbed by a bounded external disturbance vector. The addressed problem is to characterise conditions which guarantee that the instantaneous state vector or the output vector remains in a given domain defined by a set of symmetrical linear constraints. This problem is solved by imposing positive invariance conditions. These conditions also imply delay independent asymptotic stability of the associated deterministic system.

RESUME. Les systèmes à retard considérés sont représentés par un système d’équations linéaires différentielles ou aux différences avec retards. Les paramètres du système et les retards sont supposés imparfaitement connus. Le vecteur d’état instantané (pour le premier modèle) ou le vecteur de sortie (pour le second modèle) est perturbé par une perturbation additive bornée. Le problème considéré est la caractérisation de conditions garantissant que le vecteur d’état instantané ou le vecteur de sortie reste dans un domaine donné défini par un ensemble de contraintes linéaires symétriques. Ce problème est résolu en utilisant l’approche par invariance positive. Les conditions obtenues impliquent la stabilité asymptotique indépendante du retard du système déterministe associé.

KEY WORDS: linear delay systems, symmetrical constraints, positive invariance, asymptotic stability.

MOTS-CLÉS : systèmes à retard, contraintes symétriques, invariance positive, stabilité asymptotique.

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1. Introduction

Dynamical systems with delay terms are often encountered in biology, mechanics or economy. The modeling of some of these systems is not simple, so uncertainties parameters and disturbances have to be included into the model. In literature, many studies have been devoted to delay systems; in particular stability conditions for these systems have been addressed: see, e.g., [Hal77], [HIT85], [Che95], [LC95] and the references therein. In addition, delay systems are often subject to constraints, which are not generally integrated in the analysis and in the control design. This paper shows that the approach by positive invariance can be applied to delay systems. It allows to satisfy the constraints along the system trajectory and to characterise robust stability. Two models are considered: a linear delay differential model and a linear delay difference model. For each of these models, two objectives are addressed. The first objective consists in characterising the conditions which guarantee that the instantaneous state vector (for the first model) or the output vector (for the second model) respects the linear symmetrical constraints. The final objective consists in obtaining robust stability conditions which provide bounds on parameters uncertainties of delay terms. It will be shown how the resolution of the first objective via the positive invariance approach also allows to solve the last objective.

The paper is set out as follows. Section 2 presents the two models under constraints. The positive invariance results are presented in Section 3, and, in Section 4 the derived robust stability conditions are described and compared to classical conditions.

Notations. The following notations are used throughout the paper. The time-derivative and the transpose of a vector \( y(t) \) are respectively noted \( \dot{y}(t) \) and \( y(t)^T \). Matrix \( I_n \) denotes the identity matrix in \( \mathbb{R}^{n \times n} \). Vector \( 1_g \) is defined as \( 1_g = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^g \) and \( 0_g = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{R}^g \). The matrix \( \text{Diag}(\beta_i) \), for \( i = 1, \ldots, g \), denotes the diagonal matrix with components \( \beta_i \). For any matrix \( M \), \( |M| \) denotes the matrix of the absolute values of the components of matrix \( M : |M|_{ij} = |M_{ij}| \). \( \tilde{M} \) is defined by: \( \tilde{M}_{ii} = M_{ii} \) and \( \tilde{M}_{ij} = |M_{ij}| \) for \( j \neq i \). The infinity norm of a matrix \( M \in \mathbb{R}^{n \times n} \) is classically defined by: \( ||M||_\infty = \max_{i=1, \ldots, n} \sum_{j=1}^{n} |M_{ij}| \). \( C_\tau = C([-\tau, 0], \mathbb{R}^n) \) denotes the Banach space of continuous vector functions mapping the interval \( [-\tau, 0] \) into \( \mathbb{R}^n \) with the topology of uniform convergence. \( ||\varphi||_c = \sup_{-\tau \leq t \leq 0} ||\varphi(t)|| \) stands for the norm of a function \( \varphi \in C_\tau \). Furthermore, inequalities between vectors and inequalities between matrices are componentwise.

2. Problem Presentation

2.1. A Differential Model for Linear Delay Systems

A classical way to represent an autonomous delay system is through a delay differential equation [Hal77], [CGP89], [NdSDD94]:
\[
\dot{y}(t) = A_0 y(t) + \sum_{i=1}^{p} A_i y(t - \tau_i) + E w(t)
\]  

where \( y(t) \in \mathbb{R}^n \) is the instantaneous state-vector, \( w(t) \in \mathbb{R}^s \) a bounded external disturbance vector. The values of delays may be uncertain, but they are supposed to be bounded above and below by fixed positive scalars, denoted respectively \( \bar{\tau} \) and \( \underline{\tau} > 0 \):

\[
\begin{align*}
\max_{i=1,\ldots,p} \{ \tau_i \} & \leq \bar{\tau} \\
\min_{i=1,\ldots,p} \{ \tau_i \} & \geq \underline{\tau} > 0.
\end{align*}
\]

Model (1) is well-defined under the initial conditions:
- an initial time value, \( t_0, t_0 \geq 0 \),
- a continuous function \( \varphi(.) \) defined on \([-\bar{\tau}, 0]\) and belonging to \( C_T^\varphi \), the set defined \( \{ \varphi \in C_T; \| \varphi \|_\infty \leq \nu \} \). The initial trajectory of system (1) is characterised through:

\[
y(t_0 + \theta) = \varphi(\theta), \forall \theta \in [-\bar{\tau}, 0], (t_0, \varphi) \in \mathbb{R}_+ \times C_T^\varphi
\]

so that \( y(t) \equiv y_0(t) \) for \( t_0 - \bar{\tau} \leq t < t_0 \) and \( y(t) \) satisfies (1) for \( t \in [t_0, +\infty) \).

By differentiability, the solution of (1) is continuous for \( t \geq t_0 \). System (1) is supposed to have uncertain delays \( \tau_i, i = 1, \ldots, p \), and uncertain entries in \( A_i, i = 0, \ldots, p \).

### 2.2. A Difference Model for Linear Delay Systems

A natural, although not classical way to represent a delay system is through a difference equation:

\[
y(t) = \sum_{i=1}^{p} A_i y(t - \tau_i) + E w(t)
\]

where \( y(t) \in \mathbb{R}^n \) is the output-vector, \( w(t) \in \mathbb{R}^s \) a bounded external disturbance vector. The scalars \( \tau_i \), for \( i = 1, \ldots, p \) are positive. They are the system "delays". System (5) is an autonomous system for which the delays \( \tau_i, i = 1, \ldots, p \), are not very well known, but their values are bounded above and below, through relations (2)-(3). For model (5) to be well-defined, it is necessary to provide initial conditions, which can take the form (Kharitonov [Kha91]):

- of an initial time value, \( t_0 \),
- of a function \( y_0(.) \) defined on \([t_0 - \bar{\tau}, t_0]\) so that \( y(t) \equiv y_0(t) \) for \( t_0 - \bar{\tau} \leq t < t_0 \) and \( y(t) \) satisfies (5) for \( t \in [t_0, +\infty) \).

Furthermore, the solution of (5) is continuous if [Kha91]:

- function \( y_0(.) \) continuous on \([t_0 - \bar{\tau}, t_0]\),

\[
y_0(t_0) = \sum_{i=1}^{p} A_i y_0(t_0 - \tau_i) + E w(t_0).
\]

As a particular case, model (5) may be obtained by discretisation of a more classical model, as the one described in (1). But in general, delays are not supposed commensurate.
2.3. Linear Constraints

System (1) or (5) is supposed to be subject to linear constraints on \( y(t) \). These constraints are symmetrical, defined by a matrix \( G \in \mathbb{R}^{g \times n} \) and a positive vector \( \mu \in \mathbb{R}^{g} \), under the following form:

\[
-\mu \leq Gy(t) \leq \mu \quad \forall t \geq t_{0}
\]  

[6]

Constraints should be satisfied all along the system trajectories for any possible positive values of the delays \( \tau_{i}, i = 1, \ldots, p \). The symmetrical polyhedron of admissible trajectories is denoted \( S(G, \mu) \), and defined by:

\[
S(G, \mu) = \{ y \in \mathbb{R}^{n} ; \ -\mu \leq Gy \leq \mu \}.
\]  

[7]

It is assumed that instantaneous state constraints or output constraints are satisfied by the function of initial conditions, \( \varphi(.) \) or \( y_{0}(.) : \)

\[
-\mu \leq G\varphi(\theta) \leq \mu \quad \forall \theta \in [-\bar{\tau}, 0],
\]

\[
-\mu \leq Gy_{0}(t) \leq \mu \quad \forall t \in [t_{0} - \bar{\tau}, t_{0}].
\]  

[8]

Initial conditions are supposed totally arbitrary in the set \( S(G, \mu) \).

A norm boundedness condition applies to the external disturbance vector, \( w(t) \), under the form:

\[
\|GEw(t)\|_{\infty} \leq \omega, \quad \forall t \geq t_{0}.
\]  

[9]

The disturbance bound is supposed to satisfy:

\[
0 < \omega < \min_{j=1}^{q} \mu_{j}.
\]  

[10]

3. The Positive Invariance Approach

A natural way to maintain the system trajectory in \( S(G, \mu) \) is to impose the positive invariance of \( S(G, \mu) \) with respect to system (1) or (5). Many papers using this approach for classical systems can be found in literature: see, for example, [Bit91], [CH93], [BB88], [TB94]. This approach can be extended to delay systems by considering the following definition.

**Definition 1.** Positive invariance of \( S(G, \mu) \) with respect to system (1) or (5) is characterised by the following property: if the system trajectory belongs to \( S(G, \mu) \) for \( t \in [t_{0} - \bar{\tau}, t_{0}] \), then the system trajectory belongs to \( S(G, \mu) \), for any \( t \geq t_{0} \).

3.1. The Extended Farkas' Lemma

A basic tool for characterising positive invariance conditions with respect to linear dynamical systems is the extended Farkas' Lemma [Hen89]. Consider matrices \( P \in \mathbb{R}^{p \times n}, Q \in \mathbb{R}^{e \times n}, R \in \mathbb{R}^{r \times n} \), vectors \( \phi \in \mathbb{R}^{e}, \chi \in \mathbb{R}^{r}, \psi \in \mathbb{R}^{p} \). The "Extended Farkas' Lemma" can be stated as follows.
Lemma 1. [Hen89] The system of inequalities $Px \leq \psi$ is satisfied by any point of the non-empty convex polyhedral set defined by the system of constraints

$$\begin{cases} Qx & \leq \phi \\ Rx & = \chi \end{cases}$$

if and only if there exist a (dual) matrix $U \in \mathbb{R}^{p \times q}$, with non-negative coefficients and a (dual) matrix $V \in \mathbb{R}^{p \times r}$, satisfying conditions:

$$UQ + VR = P \quad [11]$$

$$U\phi + V\chi \leq \psi. \quad [12]$$

This Lemma can be considered as an extension of the well-known Farkas' Lemma [Sch87] to the matrix case.

3.2. The Case of Delay Differential Systems

From Definition 1, the positive invariance of $S(G, \mu)$ corresponds to the following implication:

$$\forall t \geq t_0, \forall \theta \in [-\tau, 0], \quad -\mu \leq Gy(t + \theta) \leq \mu \quad \implies -\mu \leq Gy(s) \leq \mu, \quad \forall s \geq t. \quad [13]$$

Under assumption (10) on the domain of admissible perturbations, positive invariance of the polyhedron $S(G, \mu)$ with respect to system (1) can be characterised as follows.

**Proposition 1.** A necessary and sufficient condition for positive invariance of $S(G, \mu)$ with respect to system (1) for any external disturbance $w(t)$ such that $\|GEw(t)\|_\infty \leq \omega$, is the existence of $p + 2$ real matrices $(H_i \in \mathbb{R}^{g \times g})$, for $i = 0, \ldots, p + 1$ such that:

$$H_iG = GA_i \quad [14]$$

$$H_{p+1}GE = GE \quad [15]$$

$$(H_0 + \sum_{i=1}^{p} |H_i|)\mu + |H_{p+1}|\omega 1_g \leq 0_g \quad [16]$$

**Sketch of the Proof.**

To facilitate the use of Lemma 1, constraints (6) defining the domain $S(G, \mu)$ can be equivalently re-written under the form:

$$Jy(t) \leq \gamma \text{ with } J = \begin{bmatrix} G \\ -G \end{bmatrix}, \quad \gamma = \begin{bmatrix} \mu \\ \mu \end{bmatrix}. \quad [17]$$

For implications (13) to be valid at any point of the admissible instantaneous state domain $S(G, \mu)$, it is necessary and sufficient to guarantee the admissibility of any
infinitesimal move starting from any point of any facet of this domain. These implications can therefore be equivalently replaced by the following ones:

\[
\begin{align*}
J y(t - \tau_k) & \leq \gamma \text{ for } k = 1, \ldots, p \\
J y(t) & \leq \gamma_j \text{ for } j \in (1, \ldots, 2g) - \{i\}, \\
J y(t) & = \gamma_i \\
\implies J i y(t) & \leq 0, \forall i = 1, \ldots, 2g, \forall t \geq t_0
\end{align*}
\]

And under assumption (9), the right assertion of implication (18) can be equivalently replaced by:

\[
J_i[A_0 y(t) + \sum_{i=1}^{p} A_i y(t - \tau_i) + E w(t)] \leq 0
\]

Then, the proof is conducted by applying Lemma 1. The complete proof can be found in [HT97].

### 3.3. The Case of Delay Difference Systems

Under the assumption (8) that initial output conditions belong to \(S(G, \mu)\), the system trajectory can be maintained in \(S(G, \mu)\) by imposing the positive invariance of \(S(G, \mu)\) with respect to system (5). So, a sequence of filtered perturbations, \(\xi = \{\xi(t)\}\) is defined by:

\[
\xi(t) = \frac{1}{\omega} G E w(t).
\]

Sequence \(\xi\) is supposed to belong to \(l_\infty\), the space of bounded sequences equipped with the norm:

\[
||\xi||_{l_\infty} = \sup_{t} ||\xi(t)||_\infty.
\]

Inequality (9) corresponds to condition \(||\xi||_{l_\infty} \leq 1\). The addressed problem is essentially similar to the one studied by Bianchini and Sznaier [BS95], but here for the delay system (5). The objective is to characterize bounded \(l_\infty\) to \(l_\infty\) operators for which the image of the bounded persistent disturbance \(\xi\) is the bounded sequence \(z = \{z(t)\}\), which should satisfy \(||z||_{l_\infty} \leq 1\).

By definition, the positive invariance of \(S(G, \mu)\) is ensured if, under the following statements:

(a) any initial trajectory \(\{y(\theta)\}\) is such that

\[-\mu \leq G y(\theta) \leq \mu \quad \forall \theta \in [t_0 - \tilde{\tau}, t_0],\]

(b) any admissible disturbance trajectory \(\{w(s)\}\) is such that \(||G E w(t)||_\infty \leq \omega\), for \(s \geq t_0\),

one gets:

\[-\mu \leq G y(t) \leq \mu, \quad \forall t \geq t_0.\]  

Under the assumption of arbitrary initial data in \(S(G, \mu)\) and under conditions (9)-(10), positive invariance of \(S(G, \mu)\) with respect to system (5) can be characterised as follows.
Proposition 2  A necessary and sufficient condition for the positive invariance of $S(G, \mu)$ relatively to system (5), for every set of positive numbers $(\tau_1, \ldots, \tau_p)$ satisfying (2), (3), is the existence of $p + 1$ real matrices $(H_i \in \mathbb{R}^{q \times q})$, for $i = 1, \ldots, p$ such that:

\begin{align*}
H_i G &= GA_i \quad [22] \\
H_{p+1} GE &= GE \quad [23] \\
\sum_{i=1}^{p} |H_i| \mu &\leq \mu - |H_{p+1}| \omega_1 \quad [24]
\end{align*}

Sketch of the Proof.

Definition (21) can be reformulated as follows, $\forall t \geq t_0$:

$$-\mathcal{M} \leq G Y_t \leq \mathcal{M} \implies -\mu \leq \Gamma A Y_t \leq \mu, \quad \forall w(t); \quad ||G E w(t)||_{\infty} \leq \omega, \quad [25]$$

with matrices $G, \Gamma, A$ and vectors $Y_t, \mathcal{M}, w$ defined by:

\begin{equation}
G = \begin{bmatrix}
G & 0 & \ldots & 0 & 0 \\
0 & G & \ldots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & G & 0 \\
0 & \ldots & \ldots & 0 & GE
\end{bmatrix} \in \mathbb{R}^{(p+1) \times (np+s)};
\end{equation}

\begin{equation}
\mathcal{M} = \begin{bmatrix}
\mu \\
\mu \\
\vdots \\
\mu \\
\omega_1 g
\end{bmatrix} \in \mathbb{R}^{(p+1)}; \quad \Gamma = [G \ldots G GE] \in \mathbb{R}^{q \times (np+s)},
\end{equation}

\begin{equation}
A = \begin{bmatrix}
A_1 & 0 & \ldots & 0 & 0 \\
0 & A_2 & \ldots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & A_p & 0 \\
0 & \ldots & \ldots & 0 & I_s
\end{bmatrix} \in \mathbb{R}^{(np+s) \times (np+s)}; \quad Y_t = \begin{bmatrix}
y(t-\tau_1) \\
y(t-\tau_2) \\
\vdots \\
y(t-\tau_p) \\
w(t)
\end{bmatrix} \in \mathbb{R}^{(np+s)}.
\end{equation}

The domain of admissible perturbations is defined by $||G E w(t)||_{\infty} \leq \omega$, or equivalently by

$$-\omega_1 g \leq G E w(t) \leq \omega_1 g.$$

Assuming that the extreme points of this domain are feasible, the implied inequalities in (25) can be equivalently replaced by:

$$-\mu \leq \Gamma A Y_t \leq \mu.$$
Initial trajectory conditions have been supposed totally arbitrary in the set $S(G, \mu)$, for $t \in [t_0 - \bar{\tau}, t_0]$ (8). So, for $t \in [t_0, t_0 + \bar{\tau}]$:

$$-M \leq GY_t \leq M \quad [28]$$

Furthermore, the lower bound on the delays, $\bar{\tau}$, has been supposed strictly positive. By recurrence, assuming that condition (28) is satisfied for $t \in [t_0, t_0 + k\bar{\tau}]$, we can prove by using the extended Farkas' Lemma condition (28) is satisfied for $t \in [t_0, t_0 + (k+1)\bar{\tau}]$. Then, we can prove that conditions (22) and (24) are both necessary and sufficient for implication (25) to be true for all $t \geq t_0$. As these conditions do not depend on the particular values of the delays (as long as they remain strictly positive), the positive invariance property is clearly "delay independent". A complete proof can be found in [HT].

3.4. Comments on the Positive Invariance Conditions

Concerning the results of Proposition 1 and Proposition 2, the following comments can be made:

- Each matrix $H_i$ can be seen as a dual matrix associated with constraint $-\mu \leq Gy(t - \tau_i) \leq \mu$ in (13) or in (21).

- Note that positive invariance conditions (14), (15) and (16) (resp. (22), (23), (24) in the difference case) do not depend on the values of $\tau_i$, for $i = 1, ..., p$. In that sense, they are delay independent.

- Note also that through definition (13) or (21) of positive invariance, the assumption of an arbitrary initial output trajectory in $S(G, \mu)$ (for $t \in [t_0 - \bar{\tau}, t_0]$) can be attached to the positive invariance property rather than to the considered system. Under this convention, this assumption is implicit in the statement of Propositions 1 and 2.

4. Stability Conditions

It is well known in classical dynamical system theory that the existence of compact positive invariant domains in the state space implies stability of the system. This property can be extended to the investigated cases of delay systems. In this way, positive invariance conditions will now be used to obtain robust stability conditions for delay systems.

4.1. The Case of Delay Differential Systems

4.1.1. Stability Conditions via Positive Invariance

The deterministic system associated with system (1) is:

$$\dot{y}(t) = A_0 y(t) + \sum_{i=1}^{p} A_i y(t - \tau_i) \quad [29]$$
Stability conditions of system (29) are now investigated. By definition, system (29) is said to be asymptotically stable if, for some vector norm, \( \| y(t) \| \rightarrow 0 \) as \( t \rightarrow \infty \). Positive invariance condition will be used under the assumptions \( G \in \mathbb{R}^{g \times n}, g \geq n \) and \( \text{rank } (G) = n \).

Robust stability attached to positive invariance of compact polyhedra appears in the following Theorem.

**Theorem 1.** If there exists: \( G \in \mathbb{R}^{g \times n}, g \geq n \) and \( \text{rank } (G) = n \), such that:

\[
\exists H_i \in \mathbb{R}^{g \times g}, \text{for } i = 0, ..., p \text{ satisfying:}
\]

\[
H_i G = GA_i \quad [30]
\]

\[
(\dot{H}_0 + \sum_{i=1}^{p} |H_i|)1_g < 0_g \quad [31]
\]

then:

(a). \( S(G,1_g) \) is positively invariant with respect to system (29),

(b). the asymptotic stability of system (29) is guaranteed independently of the values of delays \( \tau_i, i = 1, ..., p \).

**Sketch of the Proof.**

The fact that positive invariance of a compact set in the instantaneous state space implies Lyapunov stability of the system trajectories is almost trivial. Asymptotic stability of the unperturbed autonomous system (29) can be shown by construction of a strictly decreasing polyhedral Lyapunov function, of the type previously considered in [Bit91]. To this end, the following function is considered:

\[
v(y) = \max_{j=1,...,g} |(Gy)_j| = \|Gy\|_\infty. \quad [32]
\]

For rank \( (G) = n \), this function is a polyhedral norm in \( \mathbb{R}^n \), denoted \( \| . \|_G \). It is convex, but not differentiable. Consider its value at time \( t \):

\[
v(y(t)) = \| y(t) \|_G = \xi \text{ with } \xi > 0.
\]

So, by using:

\[
\frac{\partial v}{\partial t^+} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [v(y(t + \epsilon)) - v(y(t))]
\]

and by using the Razumikhin-type Theorem [Hal77], [XL94], the asymptotic stability of system (29) is proven. Relations (30), (31) derive from positive invariance conditions (14), (15) and (16) for a free choice of \( G \). Since these relations are independent of the values of the delays, asymptotic stability is obtained for any positive values of \( \tau_1, ..., \tau_p \). The complete proof can be found in [HT97].
4.1.2. Comparison with Classical Robust Stability Conditions

Some classical delay independent stability conditions for multiple delay-differential systems are established in [WCL87] and in [LA80]. The approach of Wang et al. uses the matrix measure (also called logarithmic norm) of matrix $A_0$, denoted $\nu(A_0)$, defined by:

$$
\nu(A_0) = \lim_{\epsilon \to 0^+} \frac{\|I + \epsilon A_0\| - 1}{\epsilon}
$$

[33]

The matrix measure $\nu(A_0)$ is negative if $A_0$ is asymptotically stable. Then, the stability condition given in [WCL87] for the multiple delay case is:

$$
\sum_{i=1}^{p} \|A_i\| < -\nu(A_0).
$$

[34]

This condition and expression (33) are valid for any induced matrix norm, $\| \cdot \|$. Under the choice of the infinity norm as the induced matrix norm, a condition less restrictive than (34) can be obtained by selecting a particular domain to be positively invariant. It suffices to set $G = I_n$ in Theorem 1 to obtain:

$$
(\hat{A}_0 + \sum_{i=1}^{p} |A_i|)_g < 0_g
$$

[35]

Condition (35) can be interpreted as the positive invariance condition of the particular domain $S(I_n, 1_g)$ with respect to a perturbed system (1) associated with (29).

Application of (34) in the case of uncertainties on matrices $A_i$, $i = 0, ..., p$, gives the robust stability conditions:

$$
\sum_{i=0}^{p} \|A_i\| - \nu(A_0) + \sum_{i=1}^{p} \|A_i\|
$$

[36]

In the case of the infinity norm, a condition less restrictive than (36) is obtained from the positive invariance approach under the choice $G = I_n$:

$$
\sum_{i=0}^{p} |A_i|_g < -\hat{A}_0 1_g - \sum_{i=1}^{p} |A_i|_g
$$

[37]

The general result for robust stability obtained from Theorem 1 allows a free choice of matrix $G$ with $\text{rank}(G) = n$ in conditions (30) and (31). Furthermore, asymptotic stability is preserved under condition:

$$
\sum_{i=0}^{p} |A_i|_g < -\hat{H}_0 1_g - \sum_{i=1}^{p} |H_i|_g
$$

[38]

with

$G(\Delta A_i) = (\Delta H_i)G$, $i = 0, ..., p$
Positive invariance conditions of \( S(G, \mu) \) with respect to system (1) have been given in Proposition 1. These conditions can also be specialised to the case of the compact polytope \( S(I_n, \mu) \), with \( \mu \) a positive vector in \( \mathbb{R}^n \). A sufficient delay-independent stability condition for system (29) can then be derived:

\[
\exists \text{ positive scalars } \mu_i, \ i = 1, ..., n, \text{ such that } \forall i = 1, ..., n, \ 
\mu_i(A_{0i} + \sum_{k=1}^{p} |A_{kii}|) + \sum_{j \neq i}^{p} \mu_j \sum_{k=0}^{|A_{kij}|} < 0. \tag{39}
\]

These conditions generalise the "quasi-diagonal dominance conditions" proposed, for a particular structure of delay terms, in [LA80].

**4.2. The Case of Delay Difference Systems**

**4.2.1. Stability Conditions via Positive Invariance**

The deterministic system associated with system (5) is:

\[
y(t) = \sum_{i=1}^{p} A_i y(t - \tau_i) \tag{40}
\]

The stability robustness attached to positive invariance of compact polyhedral domains can be expressed through the following Theorem.

**Theorem 2** Under the assumption that the initial trajectory of system (5) in \([t_0 - \bar{\tau}, t_0]\) is bounded, the existence of a matrix \( G, G \in \mathbb{R}^{g \times n}, g \geq n \) and rank \((G) = n\), and a scalar \(0 < \omega < 1\) such that:

\[
\exists H_i \in \mathbb{R}^{g \times g}, \text{for } i = 1, ..., p \text{ satisfying:}
\begin{align*}
H_i G &= GA_i \\
\sum_{i=1}^{p} |H_i|1_g &\leq (1 - \omega)1_g
\end{align*} \tag{41, 42}
\]

is a sufficient condition for the asymptotic stability of system (40), independently of the values of the delays \( \tau_i \), for \( i = 1, ..., p \).

**Sketch of the Proof.**

Under the assumptions that the initial output trajectory of system (5) is bounded, there exists, for any choice of matrix \( G \) with rank \((G) = n\), a scalar \( \eta \) such that:

\[
\begin{cases}
-\eta 1_g \leq Gy(t - \tau_p) \leq \eta 1_g \\
\vdots \\
-\eta 1_g \leq Gy(t - \tau_1) \leq \eta 1_g
\end{cases}, \text{ for } t \in [t_0, t_0 + \bar{\tau}] \tag{43}
\]

and

\[ ||GEw(t)||_\infty \leq \eta \omega, \text{ for } t \geq t_0. \]
Condition (42) implies:

\[
\sum_{i=1}^{p} |H_i| \eta_{1g} \leq (1 - \omega) \eta_{1g}.
\]  

[44]

This condition, together with (41), implies that the set \( S(G, \eta_{1g}) \) is positively invariant with respect to system (5). Therefore, system (5) trajectories are contained in the closed polytope \( S(G, \eta_{1g}) \), This system is Lyapunov-stable. To show asymptotic stability of the associated deterministic system (40), let us first derive from inequality (42) the following inequality, with \( \mathcal{H} = [H_1 \ldots H_p] \):

\[
\|\mathcal{H}\|_{\infty} \leq 1 - \omega \text{ with } 0 < \omega < 1.
\]

Then, this inequality implies that the spectrum of matrix \( \Pi = \sum_{i=1}^{p} H_i e^{-j\tau_i} \) lies strictly inside the unit disc of the complex plane. Now, using relations (41) for \( i = 1, \ldots, p \), matrices \( \Pi \) and \( \mathcal{N} \) satisfy \( \Pi G = G \mathcal{N} \). Thus, under condition \( \text{rank}(G) = n \), the spectrum of \( \mathcal{N} \) is included in the spectrum of \( \Pi \). It lies inside the unit disc of the complex plane, and system (29) is asymptotically stable for every positive values of \( \tau_1, \ldots, \tau_p \) satisfying (2)-(3). The complete proof can be found in [HT].

4.2.2. Comparison with Classical Robust Stability Conditions

From classical results on delay systems, the following theorem can be stated to characterise delay independent asymptotic stability of system (40).

**Theorem 3** [Hal77], [Kha91] The system (40) is delay independent asymptotically stable if and only if for all real values of \( \tau_i \) each eigenvalue of matrix

\[
\mathcal{N} = \sum_{i=1}^{p} A_i e^{-j\tau_i}
\]

[45]

lies inside the unit disc.

The following result is due to Kharitonov [Kha91]. It derives from Theorem 3, using

\[
\|\mathcal{N}\| = \|\sum_{i=1}^{p} A_i e^{-j\tau_i}\| \leq \sum_{i=1}^{p} \|A_i\|.
\]

**Proposition 3.** System (40) is delay independent asymptotically stable if

\[
\sum_{i=1}^{p} \|A_i\| < 1.
\]

[46]
In the case of the infinity norm, a less restrictive stability condition can be derived from Theorem 3:

A sufficient condition for delay independent asymptotic stability of system (40) is:

\[ \|A\|_\infty < 1, \text{ with } A = [A_1 \ldots A_p]. \]  

[47]

This stability condition (47) can be derived from the positive invariance approach by choosing \( G = I_n \). Therefore, the conditions of Theorem 2 are less restrictive since they allow a free choice of matrix \( G \) with \( \text{rank}(G) = n \).

Furthermore, the application of Proposition 3 to system

\[ y(t) = \sum_{i=1}^p (A_i + \Delta A_i)y(t - \tau_i) \]  

[48]

allows to give a bound to the admissible uncertainties on matrices \( A_i \). Stability of system (48) is guaranteed under condition

\[ \sum_{i=1}^p \|\Delta A_i\| < 1 - \sum_{i=1}^p \|A_i\| \]  

[49]

In the case of polyhedral norms, less restrictive conditions on matrices \( \Delta A_i \) can be obtained by the positive invariance approach. A direct application of stability conditions (41), (42) to the uncertain system (48) gives:

\[ \sum_{i=1}^p |\Delta H_i| + \sum_{i=1}^p |H_i| \leq 1 - \sum_{i=1}^p |H_i| \]  

[50]

with \( \Delta H_i \) defined by

\[ (\Delta H_i)G = G(\Delta A_i). \]

And positive invariance of \( S(G, 1_g) \) for the uncertain and perturbed system

\[ y(t) = \sum_{i=1}^p (A_i + \Delta A_i)y(t - \tau_i) + Ew(t) \]  

[51]

is guaranteed under condition:

\[ \omega_1 + \sum_{i=1}^p |\Delta H_i| + 1_g \leq 1 - \sum_{i=1}^p |H_i| + 1_g. \]  

[52]

Condition (49) in the case of the infinity norm is obtained as a particular instance of condition (52) for \( G = I_n \).
6. Conclusion

This note has addressed the problem of robust stability for a perturbed delay system represented by two types of models: a delay-differential equation or an input-output difference equation, with an additive bounded external disturbance vector and linear symmetrical constraints on the instantaneous state vector (in the first case) or on the output vector (in the second case). The objective was to determine conditions guaranteeing both the asymptotic stability and the respect of constraints (on the instantaneous state vector or on the output vector). The proposed methodology also applies when system parameters and delays are not very well known.

The proposed conditions allow to derive some control techniques. In [HT97] for a differential model or in [HT] for a difference model, some control design techniques have been given: the proposed control laws use the available information on the values of the delays and confer the closed-loop system with robust stability properties. This concern distinguishes these control laws from many memoryless control schemes proposed (see, e.g., [KTB94], [NdSDD94]).

7. References


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