A unified framework for linear constrained regulation problems

Jean-Claude HENNET and Eugênio B. CASTELAN

Laboratoire d’Automatique et d’Analyse des Systèmes du C.N.R.S., 7 avenue du Colonel Roche, 31077 Toulouse FRANCE

Abstract

This paper presents some results on the ability to control time-invariant linear systems by linear state feedback under some constraints on the state vector or on the control vector. The proposed unified framework provides solutions for continuous-time systems as well as discrete-time systems. This approach relies on the positive invariance properties of some domains of the state-space for stable closed-loop systems.

1. INTRODUCTION

Consider a linear dynamical system controlled by state-feedback:

\[ p[x_t] = (A + BF)x_t \text{ with } x_t \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, F \in \mathbb{R}^{m \times n}, m \leq n. \]  

(1)

\( p \) is the derivative operator in the continuous-time case and the advance operator in the discrete-time case.

A non-empty set \( \Omega \) is a positively invariant set of system (1) if and only if the state trajectory remains in \( \Omega \) for any initial state in \( \Omega \). Asymptotical stability of a linear system implies the existence of positively invariant domains. And if we are able to construct a Lyapunov function, \( v(x) \), for system (1), we can also construct positively invariant domains for this system (R.E.Kalman - J.E.Bertram 1960 [7]). These domains are simply described by the constraint: \( v(x) \leq \tau \) for some positive number \( \tau \). Consequently, the shape of the invariant domains directly depends on the selected Lyapunov function.

An efficient way to regulate a constrained system is to impose the positive invariance of the domain generated by the constraints. Then, if the invariant domain is unbounded, additional stability requirements generally have to be added to design the regulator. Polyhedral domains are naturally associated to linearly constrained control problems.

In particular, let us now assume that the state vector, \( x_t \) is subject to symmetrical linear constraints (by convention, the inequalities between vectors are componentwise):

\[ -g \leq Gx_t \leq g \text{ with } G \in \mathbb{R}^{r \times n}, \text{ rank}G = r \leq n, g \in \mathbb{R}^r, g_i > 0, i = 1, .., r. \]  

(2)

These constraints are supposed to be satisfied by the initial state of the system, \( x_0 \).
They generate the polyhedral domain $R(G, g)$ defined by:

$$R(G, g) = \{ x \in \mathbb{R}^n ; \; -g \leq Gx \leq g \}$$

(3)

If we can construct a stabilizing state-feedback gain-matrix $F$ such that the polyhedron $R(G, g)$ is positively invariant for the closed-loop system (1), then the regulation law guarantees the respect of constraints (2) for any initial state in $R(G, g)$ . Note that under a state-feedback regulation law, $u_t = Fx_t$, input constraints of the type:

$$-s \leq Su_t \leq s \text{ with } S \in \mathbb{R}^{cm}, \; c \leq m, \; \text{rank}S = c; \; s \in \mathbb{R}^c_+,$$

(4)

also define a polyhedron in the state space, $R(SF, s)$. This linear constrained regulation problem can thus be solved by imposing the closed-loop positive invariance of $R(SF, s)$.

2. EXISTENCE CONDITIONS OF INVARIANT REGULATORS


A necessary and sufficient condition for $R(G, g)$ to be a positively invariant set of system (1) is the existence of a matrix $H \in \mathbb{R}^{rr}$ and of a scalar $\alpha$ such that:

$$HG = G(A + BF)$$

(5)

$$|K|g \leq \alpha g$$

(6)

with $K = H$, $0 < \alpha \leq 1$ in the discrete-time case, $K = (\alpha I_r + H), \alpha > 0$ in the continuous-time case. (The notation $|M|$ stands for the matrix of the absolute values of the components of matrix $M$.)

Consider the subspace $\text{Ker} \; G = \{ x \in \mathbb{R}^n ; \; Gx = 0 \}$. Under relation (5), $Gx = 0$ implies $G(A + BF)x = 0$. And it is not difficult to show the following property (J.C.Hennet, E.B.Castelan 1991 [5]):

A necessary condition for the invariance of $R(G, g)$ is that the subspace $\text{Ker} \; G$ should be $(A + BF)$-invariant in the sense of Wonham [13].

By definition, if $\text{Ker} \; G$ is $(A,B)$-invariant and matrix $F$ a ”friend” of $\text{Ker} \; G$ , then $\text{Ker} \; G$ is $(A + BF)$–invariant. Matrix $G$ can be interpreted as a canonical projection matrix from $\mathbb{R}^n$ onto $(\mathbb{R}^n/\text{Ker} \; G)$ . Matrix $H$ characterizes the map induced in $(\mathbb{R}^n/\text{Ker} \; G)$ by $(A + BF)$. $y_t = Gx_t$ is the canonical projection of $x_t$ onto $(\mathbb{R}^n/\text{Ker} \; G)$ . Its evolution is described by:

$$p[y_t] = Hy_t$$

(7)
A candidate Lyapunov function for this system is the positive definite norm:

\[ w(y) = \max_{i=1}^{r} \frac{|(y)_{i}|}{g_{i}}. \]

Assume the existence of a scalar \( \alpha \), with \( 0 < \alpha \leq 1 \) in the discrete-time case, \( \alpha > 0 \) in the continuous-time case, such that relation (6) is satisfied. Then, \( w(y_{t}) \) is strictly decreasing along any trajectory of system (7), and this system is asymptotically stable.

In the original state-space, \( \mathbb{R}^{n} \), the associated function

\[ v(x) = \max_{i=1}^{r} \frac{|(Gx)_{i}|}{g_{i}} \tag{8} \]

is only positive semi-definite if \( r < n \). In general, it cannot be selected as a Lyapunov function for system (1). But we can still note that under condition (6), \( v(x) \) is non-increasing along any trajectory of system (1). And the following result can be shown:

A necessary and sufficient condition for \( R(G, g) \) to be a positively invariant set of system (1) is that the function \( v(x) \) (of relation (8)) is non-increasing along any trajectory of system (1).

Under this condition on function \( v(x) \), any domain defined by \( v(x) \leq \tau g \) for any positive scalar \( \tau \), is also a positively invariant set of system (1).

3. A STRUCTURAL INTERPRETATION OF THE INVARIANCE CONDITIONS

Let us now interpret the current vector \( y_{t} = Gx_{t} \) as an output vector. We can then introduce the system matrix (H.H.Rosenbrock 1970 [11]):

\[ P(\lambda) = \begin{bmatrix} \lambda I - A & -B \\ G & 0_{r \times m} \end{bmatrix} \tag{9} \]

The \((A+BF)\)-invariance of \( \text{Ker} \ G \) requires the placement of \( n - r \) eigenvectors of the closed-loop system in the kernel of \( G \). This is possible if and only if the equation

\[ P(\lambda_{i}) \begin{bmatrix} v_{i} \\ w_{i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10} \]

has \( n - r \) solutions \((\lambda_{i}, v_{i})\), with vectors \( v_{i} \) independent.

If the system matrix is square and regular, the set of complex numbers \( \lambda_{i} \) for which there exist vectors \( v_{i} \in \mathbb{C}^{n} \) and \( w_{i} \in \mathbb{C}^{m} \) satisfying equation (10) are called the zeros of \( S(A, B, G) \). Vectors \( v_{i} \) are the associated zero-directions (or state zero-directions) and vectors \( w_{i} \) are often called the input zero-directions.

The problem of the existence and of the number of zeros of a linear continuous-time or discrete-time system is solved in the literature (for a survey of the main results, see A.G.J. Mac Farlane and N.Karcanias 1976 [9]).
The possible cases can be outlined as follows:

1. If \( r = n \), then \( \ker G = \{0\} \). The \((A,B)\)-invariance of \( \ker G \) is automatically satisfied.
2. If \( m < r < n \), in general the system has less than \( n - r \) zeros and therefore the subspace spanned by the zero-directions is strictly included in \( \ker G \). Positive invariance of \( R(G,g) \) cannot be obtained if the number of zeros is smaller than \( n - r \).
3. The particular case when the system has \( n - r \) zeros is similar to the following case.

4. If \( r < m \), equation (10) has solutions for any complex value \( \lambda_i \). These ”controllable” solutions generate the maximal controllability subspace of \((A,B)\) included in \( \ker G \).

The three last cases can be summarized as follows: \( \text{if } r \leq m \text{ and } r < n, \text{ condition } \text{rank}(GB) = r \text{ is sufficient for the } (A,B)\text{-invariance of } \ker G \). In order to obtain the \((A+BF)\)-invariance of \( \ker G \) and closed-loop stability will not be simultaneously obtained.

If \( \text{rank}(GB) = r \leq m \), a set of independent eigenvectors spanning \( \ker G \) can be constructed by the following technique:

**1. Selection of \( n - r \) appropriate closed-loop poles:**

\((A,B)\)-invariance of \( \ker G \) requires equation (10) to admit at least \((n - r)\) solutions \((\lambda_i, v_i, w_i)\), with the set of vectors \( v_i \) for \( i = 1, \ldots, n - r \) independent. They constitute a basis of \( \ker G \) and define matrix \( V_1 = (v_1, \ldots, v_{n-r}) \) such that:

\[
GV_1 = 0. \tag{11}
\]

Then, by state feedback, \( n - r \) poles of the closed-loop system should be located at these solutions of the zero equation (10). The associated generalized eigenvectors are the column-vectors of matrix \( V_1 \) and satisfy:

\[
(A + BF)V_1 = V_1J_1 \tag{12}
\]

where \( J_1 \) is the complex Jordan form associated to the restriction of \((A+BF)\) to \( \ker G \).

If \( S(A,B,G) \) admits \( p \) invariant zeros (not necessarily distinct), this set of \( p \) complex values has to be included in the set of selected closed-loop poles. If \( p < n - r \), the \( n - p - r \) remaining closed-loop poles can be selected as desired. These \( n - r \) values, \( \lambda_i \), correspond...
to \( n - r \) solutions, \((\lambda_i, z_i)\), with the set of vectors \((z_i)\), \(i = 1, \ldots, n - r\) constituting a basis of \( \mathbb{C}^{n-r}\), of the equation:

\[
[\lambda_i N M - N A M]z_i = 0
\]  

(13)

Matrices \( M \in \mathbb{R}^{s \times (n-s)} \) and \( N \in \mathbb{R}^{(n-m) \times n} \), satisfy the following relations (Kouvaritakis and Mac Farlane, 1976):

\[
GM = 0_{s,n-s} \; ; \; NB = 0_{n-m,m} \; ; \; NM = [I_{n-m} \mid 0_{n-m,m-r}]
\]  

(14)

Any vector \( v_i \in \mathbb{C}^n \) such that \((\lambda_i, v_i)\) is a solution of relation (10) is uniquely defined by the vector \( z_i \in \mathbb{C}^{n-s} \) such that \( v_i = Mz_i \) and such that \((\lambda_i, z_i)\) is a solution of equation (13). The polynomial matrix \([\lambda N M - N A M]\) is called the zero pencil. It completely characterizes the finite zeros and the associated zero-directions of \( S(A, B, G) \).

Using for matrix \( N A M \) the same partitionning as for \( N M \), we can write the zero pencil as follows:

\[
\lambda N M - N A M = [\lambda N_{n-m} - (N A M)\mathbf{1} \mid - (N A M)\mathbf{2}]
\]  

(15)

In the square case, \((r=m)\), the invariant zeros of \( S(A, B, G) \) are the eigenvalues of matrix \( N A M \).

In the non-square (with \( r < m \) and \( GB \) full-rank) the invariant zeros of \( S(A, B, G) \) (if any) are the “input - decoupling” zeros of the pair \((N A M)_1, (N A M)_2\).

2. Construction of a set of associated eigenvectors spanning \( \text{Ker} \ G \)

For any of these \( n - r \) selected eigenvalues, \( \lambda_i \), define the associated pole-pencil \( S(\lambda_i) \) and its kernel \( K_{\lambda_i} \):

\[
S(\lambda_i) = [\lambda_i I - A \mid -B] \; ; \; K_{\lambda_i} = \begin{pmatrix} N_{\lambda_i} \\ M_{\lambda_i} \end{pmatrix}
\]  

(16)

The transmission subspace of \( \lambda_i \) is denoted \( Tr(\lambda_i) \) and defined by:

\[
Tr(\lambda_i) = \left\{ v_i \in \mathbb{C}^n ; \; \exists \; w_i \in \mathbb{C}^m ; \; S(\lambda_i) \begin{pmatrix} v_i \\ w_i \end{pmatrix} = 0 \right\}
\]  

(17)

\( \text{Ker} \ S(\lambda_i) \) has dimension \( \geq m \) and \( Tr(\lambda) \) also has dimension \( \geq m \). For any assigned closed-loop eigenvalue \( \lambda_i \) of system (1), the corresponding assigned eigenvector, \( v_i \), must belong to \( Tr(\lambda_i) \). The associated direction in the input space satisfies \( w_i = F v_i \). Any vector \( v_i \) in the transmission subspace can be written, for some vector \( k_i \) of appropriated dimension (see B.C.Moore 1976 [10]):

\[
v_i = N_{\lambda_i} k_i, \; \text{with} \; F v_i = M_{\lambda_i} k_i
\]  

(18)

In order for a candidate eigenvector to also belong to \( \text{Ker} \ G \), it should also satisfy: \( G N_{\lambda_i} k_i = 0 \).

Any vector \( k_i \in \text{Ker}(G N_{\lambda_i}) \), \( k_i \neq 0 \) can be selected for each selected eigenvalue \( \lambda_i \). In general, the set of vectors \((v_i)\) for \( i = 1, \ldots, n - r \) will be independent, because transmission subspaces essentially do not overlap (except, in the case \( m > n/2 \), for an intersection common to all of them and included in \( \mathbb{I}m(B) \)) (see N.Karcandas, B.Kouvaritakis 1978 [8]).
Independence can be checked a-posteriori and eventually restored by forbidding \( v_i \in \mathcal{I}mB \).

Equations (11), (12) remain valid when replacing the pairs of complex vectors of \( V_1 \) by the associated pairs of real vectors corresponding to the real Jordan form replacing \( J_1 \). The equivalence of these two representations allows us to indifferently consider that vectors spaces are over \( \mathbb{R} \) or \( \mathbb{C} \). The corresponding transformation is implicit.

If the pair \((A,B)\) is controllable and \( \text{Ker} \ G \) an \((A,B)\)-invariant subspace, and under the conditions \( \text{rank}(GB)=\text{rank}(G)=s \leq m \), it is possible to construct a control matrix \( F \in \mathbb{R}^{m \times n} \) such that:

(a) \( \text{Ker} \ G \) is an invariant subspace of system (1).

(b) The real generalized eigenvectors associated to the eigenvalues of \((A+BF)|_{\mathbb{R}^n/\text{Ker} \ G}\) span a subspace \( R \subset \mathbb{R}^n \) such that \( R \oplus \text{Ker} \ G = \mathbb{R}^n \). Under the controllability assumption, they can be selected as the column-vectors of a matrix \( V_2 \) satisfying:

\[
GV_2 = I_s \quad \text{with} \quad I_s \text{ the unity-matrix in } \mathbb{R}^{s \times s} \quad (19)
\]

Let \( J_2 \) be the real Jordan canonical form of \((A+BF)|_{\mathbb{R}^n/\text{Ker} \ G}\). We have:

\[
(A + BF)V_2 = V_2J_2 \quad (20)
\]

The eigenvalues of \((A + BF)|_{\mathbb{R}^n/\text{Ker} \ G}\), denoted \( \mu_i + j\sigma_i \), can be selected so as to satisfy:

\[
|\mu_i| + |\sigma_i| < 1 \quad \text{in the discrete-time case } [1] \quad (21)
\]

\[
\mu_i < -|\sigma_i| \quad \text{in the continuous-time case } [2] \quad (22)
\]

It has been shown that under such an eigenstructure assignment, matrix \( J_2 \) can be selected in place of matrix \( H \) in relation (5). Furthermore, \( |K| - \alpha I \) is an M-matrix. Therefore, there exist a positive vector \( \rho \) such that \( S(G, \rho) \) is positively invariant. Under relations (19) and (20), the left-eigenvectors of \((A + BF)|_{\mathbb{R}^n/\text{Ker} \ G}\) associated to these eigenvalues are the rows of matrix \( G \). The gain matrix \( F \) should satisfy \( GBF = J_2G - GA \). This equation has an exact solution if and only if \( \text{rank}(GB) \leq m \). And this is precisely the case if \( GB \) is full-rank, and \( r \leq m \).

If all the eigenvalues of the restriction of \((A+BF)\) to \((\mathbb{R}^n/\text{Ker} \ G)\) are simple, real and stable, relation (6) is satisfied for any positive bound vector \( \rho \), and in particular for vector \( g \). If some of these eigenvalues are complex and (or) multiple, the pole assignment should also be consistent with the values of the components of \( g \). If \( g \) is given, the eigenvalues of \((A+BF)|_{(\mathbb{R}^n/\text{Ker} \ G)}\) have to belong to some domains included in the domain defined by constraint (21) or (22) (see J.C.Hennet, J.B.Lasserre 1990 [6]). Relatively to \((\mathbb{R}^n/\text{Ker} \ G)\), the selected eigenvalues, \( \lambda_i' \), and associated directions, \( v_i' \) and \( w_i' \), satisfy:

\[
P(\lambda_i') \begin{bmatrix} v_i' & w_i' \end{bmatrix} = \begin{bmatrix} 0 \\ e_i \end{bmatrix}, \quad \text{with} \quad e_i = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} - i. \quad (23)
\]
To compute the eigenvectors ($v'_i$) in $\mathbb{R}^n$ compute the kernel of the pole pencil $S(\lambda'_i)$ for $i = 1, ..., r$, and set, for some vector $k'_i$ of appropriated dimension:

$$v'_i = N_{\lambda'_i}k'_i, \quad \text{with} \quad Fv'_i = M_{\lambda'_i}k'_i$$

(24)

The $r$ eigenvectors are selected so as to also satisfy: $GN_{\lambda'_i}k'_i = e_i$.

In particular, we can select $k_i = (GN_{\lambda'_i})^\times$ with $(GN_{\lambda'_i})(GN_{\lambda'_i})^\times = I$.

Let $V = [V_1 \mid V_2]$ be the matrix of the desired real generalized eigenvectors, and $W = [W_1 \mid W_2]$ the associated input directions. The selected real Jordan form of $(A + BF)$ is:

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}.$$ The feedback gain matrix providing the desired eigenstructure assignment is: $F = WV^{-1}$.

4. THE CASE OF INPUT CONSTRAINTS

The same approach can also be used in the case of constraints on the control vector. Under a state-feedback law, $u_t = Fx_t$, inequalities (4) can be re-written:

$$-s \leq SFx_t \leq s$$

(25)

The results of chapters 2 and 3 can be directly applied to the case of input constraints, by simply replacing matrix $G$ by matrix $SF$, and the vector of bounds $g$ by $s$. Thus, in particular, positive invariance of $R(SF, s)$ requires the $(A + BF)$–invariance of $\text{Ker} SF$. If $\text{rank}(S) = m$, this condition is equivalent to the $(A + BF)$–invariance of $\text{Ker} F$. It is interesting to note that in this case, the zeros and zero-directions determined by the system matrix $P(\lambda) = \begin{bmatrix} \lambda I - A & -B \\ SF & 0_{r \times m} \end{bmatrix}$ are respectively the open-loop eigenvalues and eigenvectors. Noting that $\text{Ker} F$ has dimension greater or equal to $n - m$, this condition imposes to keep as closed-loop eigenvalues at least $n - m$ eigenvalues of the open-loop system, with the same associated subspace spanned by the eigenvectors.

A necessary condition for the existence of an invariant (and stabilizing) state-feedback regulator in the case of $m$ independent linear constraints on the control vector is stability of at least $n - m$ open-loop eigenvalues.

In more general cases, we can assume $\text{rank}(SF) = \text{rank}(S) = c \leq m$. Then, we have $\text{Ker} F \subset \text{Ker} SF$. Condition $\text{rank}(SFB) \leq m$ being always satisfied, the $(A+BF)$-invariance of $\text{Ker} F$ can always be obtained, but the stability requirement also has to be met. At this stage of analysis, the problem of eigenstructure assignment in a complementary subspace of $\text{Ker} F$ seems to be more difficult because equation (5) takes the non-linear form (J.C.Hennet - J.P.Béziat 1990 [4]):

$$SFBF = HSF - SFA$$

(26)

But a much simpler way of constructing an invariant regulator can be obtained by inverting the roles of the state and of the control. First, note that when relation (26) is
satisfied, the control vector \( u_t \) satisfies the following evolution equation: \( Sp[ut] = HSut \).

With the control vector as the design vector, the new zero-equation can be written:

\[
\begin{bmatrix}
\lambda_i I - A & -B \\
0 & -S
\end{bmatrix}
\begin{bmatrix}
v_i \\
w_i
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] (27)

Now, as in the case of state constraints, we can set any candidate eigenvector \( v_i \) of the transmission subspace \( \text{Tr}(\lambda_i) \) and its associated input, \( w_i \), under the form:

\[
v_i = N_{\lambda_i}k_i, \quad \text{and} \quad w_i = M_{\lambda_i}k_i
\] (28)

In order for the candidate input \( w_i \) to belong to \( \text{Ker} S \), vector \( k_i \) should also satisfy: \( SM_{\lambda_i}k_i = 0 \).

A possible eigenvector assignment in \( \text{Ker} SF \) consists in:

(a) Selecting \( n - m \) stable open-loop eigenvalues, (\( \lambda_1, ..., \lambda_{n-m} \)), and their associated eigenvectors \((v_1, ..., v_{n-m})\). The associated closed loop inputs are null (\( w_i = 0 \)).

(b) Choosing \( m - c \) stable eigenvalues, (\( \lambda_{n-m+1}, ..., \lambda_{n-c} \)), and associated eigenvectors \((v_{n-m+1}, ..., v_{n-c})\) satisfying relations (28) with \( w_i \neq 0 \).

The \( n - c \) eigenvectors spanning \( \text{Ker} SF \) are the column vectors of matrix \( V_1 \). Their associated inputs are the column-vectors of \( W_1 \) (the \( n - m \) first columns of \( W_1 \) are null). The eigenvalues relatively to \( (\mathbb{R}^n/\text{Ker} SF) \) , \( \lambda_i \), are chosen in the domain bounded by constraints (21) or (22). Their associated directions, \( v'_i \in \mathbb{R}^n \) and \( w'_i \in \mathbb{R}^m \), satisfy:

\[
\begin{bmatrix}
\lambda_i' I - A & -B \\
0 & -S
\end{bmatrix}
\begin{bmatrix}
v'_i \\
w'_i
\end{bmatrix} = \begin{bmatrix} 0 \\ e_i \end{bmatrix}
\] (29)

Vectors \( e_i \), \( i = 1, ..., c \), form the canonical basis of \( \mathbb{R}^c \). For each selected complex value, \( \lambda'_i \), for \( i = 1, ..., c \), compute the kernel of the pole pencil \( S(\lambda'_i) \). and set , for some vector \( k'_i \) of appropriated dimension: \( v'_i = N_{\lambda'_i}k'_i \), \( w'_i = M_{\lambda'_i}k'_i \). Each \( k'_i \) is selected so as to satisfy: \( SM_{\lambda'_i}k'_i = e_i \).

The matrix of the desired real generalized eigenvectors in a complementary space of \( \text{Ker} SF \) is denoted \( V_2 \), and their associated input directions \( W_2 \). The feedback gain matrix providing the desired eigenstructure assignment is: \( F = [W_1 \mid W_2][V_1 \mid V_2]^{-1} \). The set of admissible initial states corresponding to this design technique is defined by: \( -s \leq SFx_0 \leq s \).

5. EXAMPLE

Consider the following data: \( A = \begin{bmatrix}
2.257 & -1.470 & -2.117 & 0.442 \\
-1.409 & 0.726 & 1.409 & -1.028 \\
1.912 & -1.470 & -1.772 & 0.442 \\
-3.731 & 2.056 & 3.731 & -1.330
\end{bmatrix} \), \( B = \begin{bmatrix}
0.910 & -0.328 \\
0.762 & 0.633 \\
0.000 & -0.723 \\
0.047 & 0.000
\end{bmatrix} \). The open-loop poles are: \( -0.302 \pm j1.028 \), \( 0.140 \), \( 0.345 \). The control constraints are defined by \( S = \begin{bmatrix}
1.0 & 0.5 \\
0.0 & 0.5
\end{bmatrix} \) and \( \rho = \begin{bmatrix}
1 \\
1
\end{bmatrix} \). The open-loop stable real
eigenvalues \( \lambda_1 = 0.140 \), \( \lambda_2 = 0.345 \) are selected as closed-loop eigenvalues, for which the associated open-loop eigenvectors span \( \text{Ker } S F \). The 2 other poles have been chosen as follows:

\[
\begin{align*}
\lambda_1' &= 0.3 + j0.6 \\
\lambda_2' &= 0.3 - j0.6
\end{align*}
\]

We then get \( A_0 = \begin{bmatrix}
3.970 & -2.891 & -3.830 & 0.733 \\
0.141 & 0.096 & -0.141 & -0.106 \\
1.819 & -1.916 & -1.679 & -0.097 \\
-3.640 & 1.993 & 3.640 & -1.302
\end{bmatrix} \cdot
\]

Using \( F = \begin{bmatrix}
1.928 & -1.339 & -1.928 & 0.5894 \\
0.128 & 0.618 & -0.128 & 0.746
\end{bmatrix} \).

On fig. 1, we can see the stable time evolution of \( Su_1 \) and \( Su_2 \), for the initial condition:

\[
x_0 = \begin{bmatrix} 0.220 & 1.304 & -0.220 & 1.524 \end{bmatrix}^T
\]

![Figure 1: Control trajectories in projection](image)

6. CONCLUSION

This paper has described an efficient approach to construct stabilizing regulators for a linear dynamical system subject to constraints on its state or on its control vector. The method consists of obtaining the positive invariance property of the admissible domain for the closed-loop system. This objective can be met under some structural conditions
through eigenstructure assignment by state-feedback. The proposed framework provides methods for solving constrained regulation problems in the continuous-time as well as in the discrete-time context. With the power of this unifying approach, a solution to the linear regulation problem under general linear constraints on the control vector is presented for the first time.

References


