Linear Programming and the Complexity of Finite-valued CSPs

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The Valued Constraint Satisfaction Problem

- Variables: $V = \{x_1, \ldots, x_n\}$
- Domain: $D$
- Cost functions: $f_i(x_{i_1}, x_{i_2}), \{x_{i_1}, x_{i_2}\} \subseteq V$

$$f_i : D \times D \to \mathbb{Q}$$

We consider only costs $\mathbb{Q}$ instead of $\mathbb{Q} \cup \{\infty\}$.

Find $\sigma : V \to D$ such that $\sum_i f_i(\sigma(x_{i_1}), \sigma(x_{i_2}))$ is minimised
Example: Min cut

$$\min \sum_{\{i,j\} \in E} f(x_i, x_j)$$

$$f(x_i, x_j) = \begin{array}{c|cc}
 x_i/x_j & 0 & 1 \\
 0 & 0 & 1 \\
 1 & 1 & 0
\end{array}$$
Example: Min cut

\[
\min \sum_{\{i,j\} \in E} f(x_i, x_j)
\]

\[
f(x_i, x_j) = \begin{array}{c|cc}
x_i / x_j & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0 \end{array}
\]
Example: Min uncut (Max cut)

\[
\min \sum_{\{i,j\} \in E} f_{mc}(x_i, x_j)
\]

\[
f_{mc}(x_i, x_j) = \begin{cases} 
0 & x_i / x_j = 0 \\
1 & x_i / x_j = 1 \\
0 & x_i / x_j = 1
\end{cases}
\]
Language-restricted VCSPs

Def. A (valued) constraint language $\Gamma$ is a set of cost functions.

Def. $\text{VCSP}(\Gamma)$ is the VCSP restricted to cost functions from $\Gamma$.

Problem

For a fixed $\Gamma$, what is the complexity of $\text{VCSP}(\Gamma)$?
A dichotomy theorem for the VCSP

Theorem (T. and Živný 2013)

For a fixed \( \Gamma \), precisely one of the following holds:

- \( \Gamma \) satisfies an algebraic condition and \( \text{VCSP}(\Gamma) \) is polynomial-time solvable; or
- \( \Gamma \) expresses the problem Max cut and \( \text{VCSP}(\Gamma) \) is \( \text{NP} \)-hard.
The algorithm

\[ x_1 \rightarrow f \rightarrow x_2 \rightarrow f \rightarrow x_3 \]
The algorithm

\[ x_1 \quad f \quad x_2 \quad f \quad x_3 \]

\[ \sigma: \quad 0 \quad 1 \quad 0 \]
The algorithm

\[ f(0, 1) \quad f(1, 0) \]
The algorithm

\[
\begin{array}{ccc}
  x_1 & f & x_2 & f & x_3 \\
  0 & 0 & 1 & 0 & 1
\end{array}
\]
The algorithm

\[ x_1 \quad f \quad x_2 \quad f \quad x_3 \]

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The algorithm

\[
\begin{align*}
(1/2) \cdot f(0,0) + (1/4) \cdot f(1,0) + (1/4) \cdot f(1,1)
\end{align*}
\]
The algorithm

\[
\frac{1}{2} f(0, 0) + \frac{1}{4} f(1, 0) + \frac{1}{4} f(1, 1)
\]
The algorithm

\[
\begin{align*}
\frac{3}{4} & \quad 0 & \quad 0 \\
0 & \quad 0 & \quad 1 \\
0 & \quad 1 & \quad 0 \\
\frac{1}{4} & \quad 1 & \quad 1 \\
\frac{1}{4} & \quad 1 & \quad \frac{3}{4}
\end{align*}
\]

\[
(\frac{1}{2}) \cdot f(0, 0) + (\frac{1}{4}) \cdot f(1, 0) + (\frac{1}{4}) \cdot f(1, 1)
\]
The algorithm

\[(3/4) \cdot f(0, 0) + (1/4) \cdot f(1, 1)\]

\[
\begin{array}{ccc}
3/4 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
1/4 & 1 & 1 \\
\end{array}
\]

\[
(1/2) \cdot f(0, 0) + (1/4) \cdot f(1, 0) + (1/4) \cdot f(1, 1)
\]
The algorithm
The algorithm
The algorithm: BLP

- The fractional problem can be formulated as a linear program, the **basic linear programming relaxation** (BLP).
- \( BLP(I) \leq Opt(I) \) for all instances \( I \) of VCSP.
- We show that if \( \Gamma \) satisfies an algebraic condition, then \( BLP(I) = Opt(I) \) for all instances \( I \) of \( VCSP(\Gamma) \).
The algorithm: BLP

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- $BLP(I) \leq Opt(I)$ for all instances $I$ of VCSP.
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The algorithm: BLP

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- $BLP(I) \leq Opt(I)$ for all instances $I$ of VCSP.
- We show that if $\Gamma$ satisfies an algebraic condition, then $BLP(I) = Opt(I)$ for all instances $I$ of VCSP($\Gamma$).
The algebraic condition

- Let \( \Omega_{sym} \) be the set of all binary and symmetric operations \( g : D \times D \to D \).

- We say that a cost function \( f \) has a (binary and symmetric) fractional polymorphism if, there exists a probability distribution \( \omega \) on \( \Omega_{sym} \) such that,

\[
\sum_{g \in \Omega_{sym}} \omega(g) \cdot f(g(x_1, y_1), g(x_2, y_2)) \leq \frac{1}{2} (f(x_1, x_2) + f(y_1, y_2)),
\]

for all \((x_1, x_2), (y_1, y_2) \in D \times D\).

- Ex. submodularity: \( \omega(\min) = \omega(\max) = \frac{1}{2} \).
A Dichotomy Theorem for the VCSP

Theorem (T. and Živný 2013)

For a fixed $\Gamma$, precisely one of the following holds:

- $\Gamma$ has a binary and symmetric fractional polymorphism and $\text{VCSP}(\Gamma)$ is polynomial-time solvable by the basic LP relaxation; or
- $\Gamma$ expresses the problem Max cut and $\text{VCSP}(\Gamma)$ is NP-hard.
Expressing Max cut

\[
\begin{array}{c|ccc}
  \frac{x}{y} & i & j \\
  \hline
  i & 1 & \ldots & 0 \\
  j & 0 & \ldots & 1 \\
\end{array}
\]

\[f_{mc}(x, y) = \frac{x}{y} \]

We say that \( \Gamma \) expresses the problem Max cut if \( f_{mc} \) can be obtained from functions in \( \Gamma \) by the following operations:

- Elimination of unnecessary domain values
- \( f'(x) := f(x, a), a \in D \) (fix a variable)
- \( f'(x, y, z) := f_1(x, y) + f_2(y, z) \) (addition)
- \( f'(x, y) := c \cdot f(x, y), c \geq 0 \) (scaling)
- \( f'(x, y) := f(x, y) + d \) (translation)
- \( f'(x) := \min_y f(x, y) \) (minimisation)
Merci pour votre attention!
The basic LP relaxation

Domain $D = \{0, 1\}$, a single binary cost function $f$.

Variables:

- $\mu_i(a)$ for $i \in V$ and $a \in \{0, 1\}$
- $\lambda_{ij}(\sigma)$ for $\{i, j\} \in E$ and $\sigma : \{x_i, x_j\} \rightarrow \{0, 1\}$

BLP:

$$\begin{align*}
\min & \quad \sum_{\{i,j\} \in E} \sum_{\sigma: \{x_i, x_j\} \rightarrow \{0,1\}} \lambda_{ij}(\sigma) \cdot f(\sigma(x_i), \sigma(x_j)) \\
\text{s.t.} & \quad \sum_{\sigma: \sigma(x_k) = a} \lambda_{ij}(\sigma) = \mu_k(a) \quad \forall k \in \{i, j\} \in E, a \in \{0, 1\} \\
& \quad \mu_i(0) + \mu_i(1) = 1 \quad \forall i \in V \\
& \quad 0 \leq \mu_i(a), \lambda_{ij}(\sigma) \leq 1
\end{align*}$$