Generic profit singularities of cyclic processes

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1. V.I. Arnold’s studies of cyclic processes

2. General model

3. Optimal strategies, equation for maximum profit

4. Singularities of profit in parametric case
V.I. Arnold’s studies in averaged optimization


Connection with classical problem in enterprize economy

\[
\frac{1}{T} \int_0^T P(u(t))dt \to \max, \quad \frac{1}{T} \int_0^T u(t)dt = \bar{u}
\]

\[
T_{u_{\min}} = T \frac{u_1 - \bar{u}}{u_1 - u_{\min}}, \quad T_{u_1} = T \frac{\bar{u} - u_{\min}}{u_1 - u_{\min}}
\]
2 General model

\[ \dot{x} = v(x, u), \quad x \in S^1, \quad u \in U, \quad v \in C^\infty, \]

\( U \) is a compact smooth manifold, \( \#U \geq 2 \).

An admissible motion is an absolute continuous map

\[ x : t \mapsto x(t) \in S^1, \quad t \in [0, T] \]

with \( \dot{x}(t) \in \text{conv}\{v(x(t), U)\} \), if \( \dot{x}(t) \) exists. In the presence of profit density \( f \in C^0 \) the problem is either

\[ \frac{1}{T} \int_0^T f(x(t))dt \to \max \quad \text{when} \quad T \to \infty; \]

or

\[ \frac{1}{T} \int_0^T e^{-\sigma t} f(x(t))dt \to \max, \]

or else

\[ \int_0^T e^{-\sigma t} f(x(t))dt / \int_0^T e^{\alpha t} dt \to \max, \]

where \( T \) is the period of a process.
Connection with search theory

On a closed path (=S¹) there is a density \( f \) of something, for examples, of berries or mushrooms, which we would like to collect by means of application of some resource. Our purpose is to maximize the effectiveness of this application. If \( \rho \) is the density of the resource applied, then the problem is

\[
\frac{\int_{S^1} f(\rho(x), x)dx}{\int_{S^1} \rho(x)dx} \rightarrow \max,
\]

where \( f(\rho(x), x) \) is the efficiency of density \( \rho(x) \) at a point \( x \), and the resource density has to satisfy

\[
\rho_{\text{min}}(x) \leq \rho(x) \leq \rho_{\text{max}}(x), \quad x \in S^1.
\]
3 Optimal strategies

An equilibrium point is a point \( x \) with \( 0 \in \text{conv}\{v(x, U)\} \).

A \( c \)-level cycle motion uses the maximum and minimum velocities when the switching function is no less or greater, respectively, then zero

\[
S(x) = f(x) - c.
\]

**Proposition 1** (V.I.Arnold, 2002) Equilibrium point and level cycle can provide maximum time averaged profit on the infinite horizon \((=\text{MAP})\).
Example 1. When the system is completely controllable or the maximum of a profit density is provided by some equilibrium point then the staying at such a point is the optimal motion.

Example 2. When all admissible velocities are positive then a level cycle could provide an optimal motion (V.I.Arnold, 2002).

Theorem 2 (H.Matos, A.D., 2004) For
- a continuous control system and
- a continuous profit density
on the circle, the maximum averaged profit on the infinite time horizon always can be provided by a level cycle or an equilibrium point.

Remark. Pontryagin maximum principle does not work here.
Theorem 3 (V.I.Arnold (2002), A.D.(2004)) For $f \in C^1$ with a finite number of critical points and a $v \in C^0$, $v > 0$, with maximum and minimum velocities coinciding at isolated points only, the MAP is the unique solution $c_0$ of equation

$$c - P(c)/T(c) = 0.$$  \hspace{1cm} (1)

Besides the left hand side of the equation is a differentiable function on $c$ near $c_0$.

Here discount rates are zero and

$$C_{\text{max}} = \max\{f(x), x \in S^1\}, \quad C_{\text{min}} = \min\{f(x), x \in S^1\},$$

$$\tan \alpha = P(c)/T(c)$$
**Theorem 4** (T. Shutkina T., A.D. (2010)) For
- \( f \in C^1 \), \( f \geq 0 \), with a finite number of critical points,
- \( v \in C^0 \), \( v > 0 \), with maximum and minimum velocities coinciding at isolated points only, and
- \( \sigma \geq 0 \)
the MAP is attended only for a one value \( c \) in the interval \([c_{\text{min}}, c_{\text{max}}]\). Besides, if this value is an interior point of the interval then it is unique solution of the equation
\[
c + \sigma P(c) + \frac{P(c)}{T(c)} = 0. \tag{2}
\]

The objective functional is another:
\[
\frac{1}{T} \int_0^T f(x(t)) dt \rightarrow \max \quad \sim \quad \frac{1}{T} \int_0^T e^{-\sigma t} f(x(t)) dt \rightarrow \max
\]

**Remark** When \( \sigma > 0 \) the case \( T \rightarrow \infty \) is trivial.
Problem reformulation

For an admissible motion $x = x(t)$ define $\rho$ :

$$\rho(x(t)) := 1/\dot{x}(t) \Rightarrow dt = \rho(x(t))dx(t).$$

The extremal problem is to maximize

$$A_\rho(f) := \int_{0}^{2\pi} e^{-\sigma \int_{0}^{x} \rho(z)dz} \int_{0}^{2\pi} f(x)\rho(x)dx/ \int_{0}^{2\pi} \rho(x)dx$$

over all measurable functions $\rho$ (=admissible densities):

$$r_1(x) \leq \rho(x) \leq r_2(x), \quad x \in S^1 \quad (= \text{constraint})$$

where $r_1 = 1/v_{max}$ and $r_2 = 1/v_{min}$.

Theorem 5 (A.D., Shutkina T.S., 2008) For a continuous profit density and a continuous positive constraint functions $r_1, r_2, r_1 \leq r_2$, there exists an admissible density $\rho_{\text{max}}$ which provides maximum time averaged profit.
Switching function

The switching function is

\[ S(x) = e^{-\sigma \int_0^x \rho(z) dz} f(x) + \sigma \int_0^x e^{-\sigma \int_0^y \rho(z) dz} f(y) \rho(y) dy + c, \]

with \( c = -\sigma P - A \) and \( P \) being the profit along the cycle.

**Proposition 6** For a differentiable profit density \( f \), continuous positive constraint functions \( r_1, r_2, r_1 \leq r_2 \), and an admissible density \( \rho \) the switching function is differentiable and has the same critical points as the density \( f \).
Structure of optimal control
Singularities of parametric optimization

Theorem 7 For a generic smooth one parameter family of pairs \((f, v > 0)\) the MAP’s germ at any parameter value is

– the germ at the origin of the function being zero for \(p \leq 0\) and one of the seven functions from the second column of Table 1 for \(p \geq 0\), and

– up to the equivalence pointed out in the third one and under the conditions from the fourth.

<table>
<thead>
<tr>
<th>№</th>
<th>Singularity</th>
<th>Equiv</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>(R^+)</td>
<td>#U \geq 2</td>
</tr>
<tr>
<td>2</td>
<td>(p^{3/2} + p^2)</td>
<td>(\Gamma_a)</td>
<td>#U \geq 2, transition through a local minimum of the profit density</td>
</tr>
<tr>
<td>3</td>
<td>(p^{3/2} - p^2)</td>
<td>(\Gamma_a)</td>
<td>#U \geq 2, transition through a local maximum of the profit density</td>
</tr>
<tr>
<td>4</td>
<td>(p^{3/2})</td>
<td>(\Gamma)</td>
<td>#U \geq 2, transition through a tangent double point of the velocity used</td>
</tr>
<tr>
<td>5</td>
<td>(p^2)</td>
<td>(R^+)</td>
<td>#U \geq 3, transition through a triple point</td>
</tr>
<tr>
<td>6</td>
<td>(p^3)</td>
<td>(R^+)</td>
<td>#U \geq 2, switching at a regular double point</td>
</tr>
<tr>
<td>7</td>
<td>(-p^{7/2})</td>
<td>(\Gamma)</td>
<td>(\dim U &gt; 0), transition through a swallow point</td>
</tr>
</tbody>
</table>

Besides, for a generic pair and any one sufficiently close to it, the graphs of such profits can be reduced one to another by a smooth \(\Gamma\)-equivalence close to the identity.
Remark 1

$R^+$-equivalence: $(x, g) \mapsto (\tilde{x} = h(x), \tilde{g} = g \circ h^{-1} + \phi)$

$\Gamma$-equivalence: $(x, g) \mapsto (\tilde{x} = h(x), \tilde{g} = k(x, g))$

$\Gamma_a$-equivalence: $(x, g) \mapsto (\tilde{x} = h(x), \tilde{g} = k(x)g + \phi)$

Singularities of extremal velocities

Generically constraint $r_2$ is up to $R^+$-equivalence either zero or one of three functions near the origin:

1) $|x|$; 2) $\max\{|x|, p\}$; 3) $\max\{-w^4 + pw^2 + xw \mid w \in R\}$.

For the constraint $r_1$ one needs to change the sign of these functions. A point with one of these singularities we will call double, triple or swallow point, respectively.

The closure of union of such singular points is called Maxwell set.
Generically this set is either empty or
- a smooth curve when the number $\#U = 2$, or
- a smooth curve with triple points when $\#U = 3$, and triple points for minimum and maximum velocities are the same, or else
- a smooth curve with triple (and swallow) points with transversal self-intersection outside them when $\#U > 3$ ($\dim U \geq 1$, respectively), and triple and swallow points for minimum and maximum velocities are different.

Points of type (1), (2), (3) lead to singularities of averaged profit if the respective velocity is used near the point.
Example

Figure 2: Singularities of square distance to the beach
Singularities of transition through extremum

Figure 3: Tangency of switching curve with a fiber
Singularities of switching

Figure 4: Transversal crossing of switching curve and Maxwell set leads to singularity 6
Stationary strategies in cyclic processes

Theorem 8 (Mena-Matos, A.D., 2004) For a generic smooth one-parameter family of pairs of control systems and profit densities on the circle and any value of the parameter admitting equilibrium points, the germ of the best averaged profit over the stationary strategies at such a value is the germ at the origin of one of the functions in the second row of Table 2. Besides, the singularities are stable.

<table>
<thead>
<tr>
<th>Type</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singularity</td>
<td>0</td>
<td></td>
<td>( p )</td>
<td>( \sqrt{p}, p &gt; 0 )</td>
<td>( \max{0, 1 + \sqrt{p}} )</td>
</tr>
<tr>
<td>Equivalence</td>
<td>( \mathbb{R}^1 )</td>
<td>( \mathbb{R}^1 )</td>
<td>( \mathbb{R}^1 )</td>
<td>( \mathbb{R}^1 )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

These singularities are well known:


Singularities of stationary domain

Theorem 9 (Mena-Matos, A.D., 2004) For a generic one-parameter family of control systems on the circle, the germ of the stationary domain at any of its boundary points is the germ at the origin of one of the seven sets from Table 3 in an appropriate smooth coordinate system foliated over the parameter. Besides,

- the number of different values of the control parameter must be no less than 2 for singularities 1 and 2, no less than 3 for singularities 3 and 4 and equal to 2 for singularity 5 and

- the stationary domains for a generic family of systems and any one sufficiently close to it can be carried one to another by a $C^\infty$-diffeomorphism that is close to the identity and preserves the natural foliation over the parameter.

<table>
<thead>
<tr>
<th>1</th>
<th>$x \leq 0$</th>
<th>$2_\pm$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5_\pm$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p \geq \pm x^2$</td>
<td>$p \leq</td>
<td>x</td>
<td>$</td>
<td>$x \geq -</td>
</tr>
</tbody>
</table>

Table 3:
Singularities of transition between strategies
A parameter value is a *transition one* if near it the MAP cannot be provided by one and only one type of strategy.

**Theorem 10 (Mena-Matos, A.D., 2004)** For a generic smooth one-parameter family of pairs of control systems and profit densities on the circle, the germ of MAP at a transition value is the germ at the origin of one of the two functions from the second column of Table 3 up to $R^+$-equivalence. Besides, transition parameter values for a generic pair and any one sufficiently close to it can be carried one to another by a diffeomorphism of the parameter space which is close to the identity and preserves the type of the transition singularity.

<table>
<thead>
<tr>
<th>No</th>
<th>Singularity</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$</td>
<td>p</td>
</tr>
<tr>
<td>2</td>
<td>$\max \left{ 0, \frac{p}{</td>
<td>\ln p</td>
</tr>
</tbody>
</table>

Here $H = h(p, \frac{p}{|\ln p|}, \frac{\ln |\ln p|}{|\ln p|})$, $h$ is smooth, $h(p, 0, 0) \equiv 0$. 

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Stability and Generalization

**Theorem 11** *(Mena-Matos, A.D., 2004)* In a generic one parameter case all singularities are stable up to small perturbation of the pair of families.

**Theorem 12** *(Mena-Matos, A.D., 2005)* The obtained results are true in the case of a one dimensional parameter for a generic family of control systems (profit densities) when a generic family of profit densities (control systems, respectively) is fixed.
Comment

For generic families of polydynamical systems and profit densities the classifications are obtained up to dimension 3 of the parameter for stationary strategies (H. Mena-Matos & C. Moreira, 2007).
References


