Bilinear control of nonlinear Schrödinger and wave equation

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Bilinear control

Model system

\[ i \partial_t \psi(t, x) + \partial_x^2 \psi(t, x) = -u(t) \mu(x) \psi(t, x). \] (1)

\( u \) (the control) and \( \mu \) the real valued potential .

So, at each time \( t \), the available control \( u(t) \) is only the amplitude and not a distributed fonction.
Bilinear control

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Aim: **local control by perturbation**
Introduction

Main results

Idea of proof

Other results

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Aim: local control by perturbation

Other results: nonlinear Schrödinger and nonlinear wave equation
Bibliography

Exact controllability

- **Negative result**: Ball-Marsden-Slemrod (82)
- **Positive result**: Local exact controllability in 1D: in $H^7$, in large time Beauchard (05), Coron (06): $T_{\text{min}} > 0$, controllability in 1D between eigenstates: Beauchard and Coron (06)
Bibliography

Approximate controllability

- By Gallerkin approximation and finite dimensional methods
  Chambrion-Mason-Sigalotti-Boscain (09)
- By stabilization Nersesyan (09)
- Exact controllability "at $T = \infty$" Nersesyan-Nersisyan (10)
First obstruction Ball-Marsden-Slemrod

Theorem (Ball-Marsden-Slemrod 82)

*If the multiplication by $\mu$ is bounded on the functional space $X$, then the set of reachable states is a countable union of compact sets of $X$ $\Rightarrow$ no controllability in $X$.)*
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Once the functional space $X$ is chosen, we must chose a potentiel $\mu$ enough regular to be able to do a perturbation theory, but not too much otherwise Ball-Marsden-Slemrod applies.
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Once the functional space $X$ is chosen, we must choose a potential $\mu$ enough regular to be able to do a perturbation theory, but not too much otherwise Ball-Marsden-Slemrod applies.

First solution given by K. Beauchard: use of Nash-Moser theorem.
Improved method (with K. Beauchard): prove directly that the system can be well posed even if the potential is "bad" $\Rightarrow$ optimal with respect to regularity and time of control; easier proof that can be extended to other cases.
Main results

Denote $\varphi_k$ the eigenfunctions of the Dirichlet Laplacian operator. We control near the ground eigenstate $\varphi_1$ with solution

$$\psi_1(t) = e^{-i\lambda_1 t} \varphi_1.$$  

$S$ is the unit sphere of $L^2(]0, 1[)$. 

**Theorem (with K. Beauchard)**

*Let $T > 0$ and $\mu \in H^3(]0, 1[, \mathbb{R})$ be such that

$$\exists c > 0 \text{ such that } \frac{c}{k^3} \leq |\langle \mu \varphi_1, \varphi_k \rangle|, \forall k \in \mathbb{N}^*.$$  

(2)*

There exists $\delta > 0$ such that for any $\psi_f \in S \cap H^3_0(]0, 1[, \mathbb{C})$ with $\|\psi_f - \psi_1(T)\|_{H^3} < \delta$ there exists a control $u \in L^2(]0, T[, \mathbb{R})$ s.t. the solution of (1) with initial condition

$$\psi(0) = \varphi_1$$

and control $u$ satisfies $\psi(T) = \psi_f$. 
Remarks about assumption (2)

\[ \langle \mu \varphi_1, \varphi_k \rangle_{L^2_x} = \frac{4[(-1)^{k+1}\mu'(1) - \mu'(0)]}{k^3\pi^2} - \frac{\sqrt{2}}{(k\pi)^3} \int_0^1 (\mu \varphi_1)'''(x) \cos(k\pi x) \, dx \]

\[ = \frac{4[(-1)^{k+1}\mu'(1) - \mu'(0)]}{k^3\pi^2} + \frac{\ell^2 \text{ sequence}}{k^3}. \]

and we can prove that assumption (2) is generic in \( H^3([0,1]) \).
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Such assumption implies that multiplication by \( \mu \) does not map \( H^3_0 \) into itself.
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Such assumption implies that multiplication by \( \mu \) does not map \( H^3_{(0)} \) into itself.

Rk : there are some cases where assumption (2) is not fulfilled but Beauchard and Coron manage to prove the controllability with additional techniques : return method or power series expansions.
"Regularizing" effect

\[ H^3_{(0)} = D \left( (-\Delta_{\text{Dirichlet}})^{3/2} \right) \]
\[ = \{ u \in H^3 \mid u(0) = u(1) = 0 = u''(0) = u''(1) \} \]

Proposition (with K. Beauchard)

Let \( f \in L^2((0, T), H^3 \cap H^1_0) \) \((\text{not necessarily } H^3_{(0)}).\) Then, the solution \( \psi \) of

\[
\begin{cases}
  i \partial_t \psi(t, x) + \partial_x^2 \psi(t, x) & = f \\
  \psi(0) & = 0
\end{cases}
\]

belongs to \( C^0([0, T], H^3_{(0)}) \)
Method of proof

- Prove that the **linearized problem is controllable** by Ingham Theorem.
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Rk: In certain cases treated by Beauchard and Coron, we can get controllability even if the linearized system is not controllable (use return method and quasi-static transformation or expansion to higher order). Our result should improve the regularity in these results.
Controllability of the linearized system

We linearize around the trajectory $\psi_1(t, x) = e^{-i\lambda_1 t} \phi_1$.

\[
\begin{aligned}
    i \partial_t \psi(t, x) + \partial_x^2 \psi(t, x) &= -v(t) \mu(x) \psi_1(t, x) \\
    \psi(0, x) &= 0.
\end{aligned}
\]
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\end{array}
\right.
\]

\[
\psi(T) = \sum_{k=1}^{\infty} i \langle \mu \varphi_1, \varphi_k \rangle \left( \int_0^T \nu(t) e^{i(\lambda_k - \lambda_1) t} dt \right) e^{-i\lambda_k T} \varphi_k.
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\[
\psi(T) = \sum_{k=1}^{\infty} i\langle \mu \varphi_1, \varphi_k \rangle \left( \int_0^T v(t)e^{i(\lambda_k - \lambda_1)t} dt \right) e^{-i\lambda_k T} \varphi_k.
\]

\( \psi(T) = \psi_f \) is equivalent to the trigonometric moment problem

\[
\int_0^T v(t)e^{i(\lambda_k - \lambda_1)t} dt = d_{k-1}(\psi_f) := \frac{\langle \psi_f, \varphi_k \rangle e^{i\lambda_k T}}{i\langle \mu \varphi_1, \varphi_k \rangle}, \forall k \in \mathbb{N}^*.
\] (3)

By Ingham theorem : \( \forall T > 0; \psi_f \in H^3_{(0)}(]0, 1[ \) there exists one \( v \in L^2(]0, T[) \) solution. (if \( T = 2/\pi \), it is only Fourier series in time)
Ingham Theorem

Theorem (Ingham, Haraux)

Let \( N \in \mathbb{N}, (\omega_k)_{k \in \mathbb{Z}} \) be an increasing sequence of real numbers such that

\[
\omega_{k+1} - \omega_k \geq \gamma > 0, \forall k \in \mathbb{Z}, |k| \geq N,
\]

\[
\omega_{k+1} - \omega_k \geq \rho > 0, \forall k \in \mathbb{Z},
\]

and \( T > 2\pi/\gamma \). The map

\[
J : F := \text{Clos}_{L^2([0, T])}(\text{Span}\{e^{i\omega_k t}; k \in \mathbb{Z}\}) \rightarrow l^2(\mathbb{Z}, \mathbb{C})
\]

\[\n \nu \mapsto \left(\int_0^T \nu(t)e^{i\omega_k t}dt\right)_{k \in \mathbb{Z}}\]

is an isomorphism.

This is a kind of Fourier decomposition for "not exactly orthogonal basis" (Riesz basis).
Proof of the "regularizing" effect

\[
\int_0^t e^{-i\partial_x^2 s} f(s) ds = \sum_{k=1}^{\infty} \left( \int_0^t \langle f(s), \phi_k \rangle_{L_x^2} e^{i\lambda_s} ds \right) \phi_k = \sum_{k=1}^{\infty} x_k(t) \phi_k.
\]
Proof of the "regularizing" effect

\[ \int_0^t e^{-i\partial_x^2 s} f(s) ds = \sum_{k=1}^{\infty} \left( \int_0^t \langle f(s), \varphi_k \rangle_{L^2_x} e^{i\lambda_k s} ds \right) \varphi_k = \sum_{k=1}^{\infty} x_k(t)\varphi_k. \]

We need to estimate \( \|x_k(t)\|_{h^3}^2 = \sum_{k=1}^{\infty} |k^3 x_k(t)|^2 \).
Proof of the "regularizing" effect

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\]

We need to estimate \( \| x_k(t) \|_{h^3}^2 = \sum_{k=1}^{\infty} |k^3 x_k(t)|^2 \)

\[
\langle f(s), \varphi_k \rangle_{L_x^2} = \int_0^1 f(s, x) \sin(k\pi x) dx
\]

\[
\quad = -\frac{1}{(k\pi)^2} \int_0^1 f''(s, x) \sin(k\pi x) dx
\]

\[
\quad = \frac{1}{(k\pi)^3} \left( (-1)^k f''(s, 1) - f''(s, 0) \right)
\]

\[
\quad - \frac{1}{(k\pi)^3} \int_0^1 f'''(s, x) \cos(k\pi x) dx.
\]
Proof of the "regularizing" effect

\[ \| x_k(t) \|_{H^3}^2 \lesssim C \sum_{k=1}^{\infty} \left| \int_0^t f''(s, 1)e^{i\lambda_k s} ds \right|^2 + \text{idem} \]

\[ + \sum_{k=1}^{\infty} \left| \int_0^t \int_0^1 f'''(s, x) \cos(k\pi x)e^{i\lambda_k s} dx ds \right|^2 \]

\[ \lesssim C \left\| f''(., 1) \right\|_{L^2([0, 2/\pi])} + \text{idem} + t \left\| f''' \right\|_{L^2([0, T], L^2)} \]

from Plancherel (in time) formula on \([0, 2/\pi]\) (first estimate) and Cauchy Schwartz (second estimate).
Other results

The method is quite robust and can be applied to other problems:

- **Nonlinear Schödinger equation** near constant in space solution
- **Linear and nonlinear wave equation** near constant solution
Control smoother data with smoother control

Theorem (with K. Beauchard)

Let $T > 0$ and $\mu \in H^5(0,1, \mathbb{R})$ satisfying (2) There exists $\delta > 0$ such that for any $\psi_f \in S \cap H^5_0(0,1, \mathbb{C})$ with $\|\psi_f - \psi_1(T)\|_{H^5} < \delta$ there exists a control $u \in H^1_0(0, T, \mathbb{R})$ s.t. the solution of (1) with initial condition

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Rq : Actually, we prove that the solution fulfills

$$\partial^2_x \psi + u(t) \mu \psi \in C^0([0, T], H^3_0).$$

Therefore, $\psi(t)$ does not, in general, belong to $H^5_0([0, 1])$ for $t \in (0, T)$ (OK if $u(t) = 0$).
3D ball with radial data

We prove similar results for the linear Schrödinger equation on the 3D ball with radial data: same eigenvalues and behavior is "one dimensional".
Control of nonlinear Schrödinger equation

Nonlinear Schrödinger equation on \( ]0, 1[ \) with Neumann boundary conditions

\[
\begin{align*}
\left\{ \begin{array}{ll}
    i \frac{\partial \psi}{\partial t}(t, x) &= -\frac{\partial^2 \psi}{\partial x^2}(t, x) + |\psi|^2 \psi(t, x) - u(t) \mu(x) \psi(t, x) \\
    \frac{\partial \psi}{\partial x}(t, 0) &= \frac{\partial \psi}{\partial x}(t, 1) = 0.
\end{array} \right.
\] (4)

We control around the trajectory \( \psi(t) = e^{-it} \)

**Theorem (with K. Beauchard)**

Let \( T > 0 \) and \( \mu \in H^2(0, 1) \) be such that

\[
\exists c > 0 \text{ such that } \left| \int_0^1 \mu(x) \cos(k\pi x) \, dx \right| \geq \frac{c}{\max\{1, k\}^2}, \forall k \in \mathbb{N}. \] (5)

There exist \( \delta > 0 \) such that for any \( \psi_f \in S \cap H^2_{(0,N)}(]0, 1[, \mathbb{C}) \) with \( \|\psi_f - e^{-iT}\|_{H^2} < \delta \) there exists a control \( u \in L^2(]0, T[, \mathbb{R}) \) s.t. the solution of (4) with initial condition \( \psi(0) = \varphi_1 \) and control \( u \) satisfies \( \psi(T) = \psi_f \).
Nonlinear wave equations

Nonlinear wave equation on \([0, 1]\) with Neumann boundary conditions

\[
\begin{aligned}
    w_{tt} &= w_{xx} + f(w, w_t) + u(t)\mu(x)(w + w_t) \\
    w_x(t, 0) &= w_x(t, 1) = 0,
\end{aligned}
\]

We assume \(f \in C^3(\mathbb{R}^2, \mathbb{R})\) such that \(f(1, 0) = 0\) (the constant \(w \equiv 1\) is solution) and \(\nabla f(1, 0) = 0\) (the linearized around 1 is the linear wave equation).

**Theorem**

Let \(T > 2\), \(\mu \in H^2((0, 1), \mathbb{R})\) be such that (5) holds. There exists \(\delta > 0\) such that for any \((w_f, \dot{w}_f) \in H^3_{(0, N)} \times H^2_{(0, N)}(]0, 1[, \mathbb{R})\) with

\[
\|w_f - 1\|_{H^3} + \|\dot{w}_f\|_{H^2} < \eta \text{ there exists a control } u \in L^2(]0, T[, \mathbb{R}) \text{ s.t. the solution of (6) with initial data } (w, w_t)(0, x) = (1, 0) \text{ and control } u \text{ satisfies } (w, w_t)(T) = (w_f, \dot{w}_f).\]
Further problems

- Higher dimensions: but the spectral gap used to apply Ingham theorem is no more guaranteed.
- May be some negative results more precise than Ball-Marsden-Slemrod using microlocal analysis
THANK YOU FOR YOUR ATTENTION !!!!!