

# Tableau calculi for $\mathcal{CSL}$ over minspaces

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**Abstract** The logic of comparative concept similarity  $\mathcal{CSL}$  has been introduced in 2005 by Shemeret, Tishkovsky, Zakharyashev and Wolter in order to express a kind of qualitative similarity reasoning about concepts in ontologies. The semantics of the logic is defined in terms of distance spaces; however it can be equivalently reformulated in terms of preferential structures, similar to those ones of conditional logics. In this paper we consider  $\mathcal{CSL}$  interpreted over symmetric and non-symmetric distance models satisfying the limit assumption, the so-called minspace distance models. We contribute to automated deduction for  $\mathcal{CSL}$  in two ways. First we prove by the finite filtration method that the logic has the effective finite model property with respect to its preferential semantics. Then we present a decision procedure in the form of a labelled tableau calculus for both cases of  $\mathcal{CSL}$  interpreted over symmetric and non-symmetric minspace distance models. The termination of the calculus is obtained by imposing suitable blocking conditions.

## 1 Introduction

The logics of comparative concept similarity  $\mathcal{CSL}$  has been introduced in [8] to capture a form of reasoning about qualitative comparison between concept instances. In these logics we can express assertions or judgments of the form: “Renault Clio is more similar to Peugeot 207 than to Ferrari 430”. These logics may find an application in ontology languages, whose logical base is provided by Description Logics, allowing concept definitions based on proximity/similarity measures. For instance, the color “Reddish” may be defined as a color which is more similar to a prototypical “Red” than to any other color [8] (in some color model as RGB). The aim is to dispose of a language where logical concept classification provided by standard DL is integrated with classification mechanisms based on calculation of proximity measures, typical for instance of domains like bioinformatics or linguistic. In this context, several languages comprising both absolute similarity measures and comparative similarity operator(s) have been considered [9]. In this paper we concentrate on the logic  $\mathcal{CSL}$  which is obtained from the Boolean logic by adding one binary connective  $\Leftarrow$  expressing comparative similarity to a propositional language. In this language the above examples can be encoded (using a description logic notation) by:

- (1) Reddish  $\equiv$  {Red}  $\Leftarrow$  {Green, ..., Black}
- (2) Clio  $\sqsubseteq$  (Peugeot207  $\Leftarrow$  Ferrari430)

Comparative similarity assertions such as (2) might not necessarily be the result of an objective numerical calculation of similarity measures, but they could be

determined by the (integration of) subjective opinions of agents, answering, for instance, to questions like: “Is Clio more similar to Peugeot207 or to Ferrari 430?”. In a more general setting, the language might contain several connectives  $\Leftarrow_{\text{Feature}}$  corresponding to a specific distance function  $d_{\text{Feature}}$  measuring the similarity of objects with respect to each Feature (size, price, power, taste, color...).

The semantics of  $\mathcal{CSL}$  is defined in terms of distance spaces, that is to say structures equipped by a distance function  $d$ , whose properties may vary according to the logic under consideration. In this setting, the evaluation of  $A \Leftarrow B$  can be informally stated as follows:  $x \in (A \Leftarrow B)$  iff  $d(x, A) < d(x, B)$  meaning that the object  $x$  is an instance of the formula  $A \Leftarrow B$  (i.e. it is more similar to  $A$  than to  $B$ ) if  $x$  is strictly closer to  $A$ -objects than to  $B$ -objects according to the distance function  $d$ , where the distance of an object to a set of objects is defined as the *infimum* of the distances to each object in the set.

Properties of  $\mathcal{CSL}$  with respect to different classes of models have been investigated in [8,4,9,10]. Moreover,  $\mathcal{CSL}$  over arbitrary distance spaces can be seen as a fragment, indeed a powerful one (including for instance the logic  $\mathbf{S4}_u$  of topological spaces), of a general logic for spatial reasoning comprising different modal operators defined by (bounded) quantified distance expressions (namely the logic  $\mathcal{QML}$  [10]). The satisfiability problem for  $\mathcal{CSL}$  (and in particular in the case of symmetric minspaces) is EXPTIME-complete. Finally, when interpreted over subspaces of  $\mathbb{N}^n$ ,  $\mathbb{Z}^n$  or  $\mathbb{R}^n$ , it turns out that this logic is undecidable.

In this paper we consider the semantics of  $\mathcal{CSL}$  induced by *minspaces*, that is to say distance spaces where the infimum of a set of distances is actually their *minimum* (the so-called limit assumption property). In this case, the logic  $\mathcal{CSL}$  is naturally related to some conditional logics, whose semantics is often expressed in terms of preferential structures: that is to say possible-world structures equipped by a family of strict (pre)-orders  $\prec_x$  indexed on objects/worlds [6].

Moreover, it is shown that, under the limit assumption,  $\mathcal{CSL}$  is able to distinguish between validity in symmetric and non-symmetric models<sup>1</sup> In this work we consider the logic  $\mathcal{CSL}$  as defined over symmetric and respectively non-symmetric minspaces. The minspace property entails the restriction to spaces where the distance function is discrete. This requirement does not seem incompatible with the purpose of representing qualitative similarity comparisons, whereas it might not be reasonable for applications of  $\mathcal{CSL}$  to spatial reasoning. The distinction between the symmetric and non-symmetric case is significant, for instance a kind of circular KB containing:

$$\begin{aligned} \text{Clio} &\sqsubseteq (\text{Golf} \Leftarrow \text{Ferrari430}) \\ \text{Golf} &\sqsubseteq (\text{Ferrari430} \Leftarrow \text{Clio}) \\ \text{Ferrari430} &\sqsubseteq (\text{Clio} \Leftarrow \text{Golf}) \end{aligned}$$

is satisfiable in non-symmetric minspace models with non-empty concepts Clio, Golf, Ferrari430, whereas it is not in symmetric minspace models [8].

<sup>1</sup> In contrast, in the general case where limit assumption is not assumed the logic cannot distinguish between symmetric and non-symmetric models.

In this paper we contribute to automated deduction of  $\mathcal{CSL}$  over minspaces. We begin by showing that the semantics of  $\mathcal{CSL}$  over minspaces can be equivalently restated in terms of preferential models satisfying some additional conditions, namely modularity, centering, and limit assumption. For the symmetric case, preferential models involve a four-place relation, or more intuitively a relation between pairs of objects (Pair models). In the non-symmetric case, the corresponding preferential semantics is formulated in terms of a family of pre-orders (or a ternary relation) as described in [2].

We prove the effective finite model property of the logic with respect to the pair-model semantics. According to this, any model of a formula gives rise to a *finite* model whose size is bounded by a computable function of the length of the formula. Decidability and EXPTIME upper bound complexity of the logic are straightforward consequences of this result. The result is proved by a finite filtration method which is a standard method in modal logic. The filtration method proved to be very powerful for establishing decidability of many logics including logics outside the family of modal and description logics for which it was originally developed. Very recently the filtration technique has been shown strongly related to tableau blocking mechanisms used for detecting loops in tableau derivations, thereby obtaining termination of tableau algorithms [7].

Next we define a tableau calculus for checking satisfiability of  $\mathcal{CSL}$  formulas in pair models. Our tableau procedure makes use of labelled formulas and of positive and negative preference relation. To the best of our knowledge it provides the first practically-implementable decision procedure for  $\mathcal{CSL}$  logic over symmetric minspaces. The calculus makes use of blocking conditions to obtain termination. But the blocking conditions are not only an expedient to obtain termination, they have also a semantical meaning: they prevent the potential generation of infinite models violating the limit assumption. For this reason the soundness of the calculus under the termination restrictions is not a trivial consequence of the soundness of the tableau rules, as it discards some candidate models. The soundness argument relies on a partial filtration method, strongly related to the filtration method mentioned above. The whole argument has some similarity with *unrestricted blocking* technique presented in [7].

We finally we adapt our calculus to the simpler non-symmetric case. For this case, a tableau calculus was presented in [2], the calculus presented here is both significantly different and conceptually simpler.

## 2 Syntax and Semantics

$\mathcal{CSL}$  is defined as a propositional language. Formulas can be also be viewed as concepts expressions in a description logic notation that we like to adopt here. Propositional variables (or concept names) are denoted by  $P_1, P_2, \dots, P_n$  and the logical connectives are  $\neg, \sqcup, \sqcap$ , and  $\Leftarrow$  (*comparative similarity operator*). *Formulas* are defined as follows:

$$C, D \stackrel{\text{def}}{=} P_i \mid \neg C \mid C \sqcup D \mid C \sqcap D \mid C \Leftarrow D.$$

Moreover we use the truth constants  $\top \stackrel{\text{def}}{=} P \sqcup \neg P$  and  $\perp \stackrel{\text{def}}{=} P \sqcap \neg P$ , for some propositional variable  $P$ .

The original semantics of  $\mathcal{CSL}$  [8] makes use of *distance spaces* to encode similarity measures between objects. In this paper, we restrict ourselves to symmetric distances spaces, which are presented below.

We say that a pair  $(\Delta, d)$  where  $\Delta$  is a non empty set and  $d$  is a function from  $\Delta$  to  $\mathbb{R}^{\geq 0}$  is a *symmetric distance space* if  $d$  satisfies the following conditions<sup>2</sup>:

$$\begin{aligned} (\text{id}) \quad & d(x, y) = 0 \iff x = y \\ (\text{sym}) \quad & d(x, y) = d(y, x) \end{aligned}$$

We define  $d(C, D) \stackrel{\text{def}}{=} \inf\{d(x, y) \mid x \in C, y \in D\}$ , for all non-empty subsets<sup>3</sup>  $C, D$  of  $\Delta$ . We call  $(\Delta, d)$  a *symmetric minspace* if it additionally satisfies the following condition on the existence of a minimum of a set of distances:

$$(\text{min}) \quad C \neq \emptyset \text{ and } D \neq \emptyset \Rightarrow \exists x_0 \in C \exists y_0 \in D d(x_0, y_0) = d(C, D).$$

$\mathcal{CSL}$ -symmetric minspace models (or just minspace models, since we consider here only the symmetric ones) are then defined as Kripke models based on a minspace.

**Definition 1 (Minspace model).** A minspace (distance) model  $\mathcal{I}$  is a triple  $\mathcal{I} = \langle \Delta^{\mathcal{I}}, d^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$  where:

- $(\Delta^{\mathcal{I}}, d^{\mathcal{I}})$  is a (symmetric distance) minspace.
- $\cdot^{\mathcal{I}}$  is the evaluation function defined as usual on propositional variables and boolean connectives, and as follows for the concept similarity operator:

$$(A \Leftarrow B)^{\mathcal{I}} \stackrel{\text{def}}{=} \{x \in \Delta^{\mathcal{I}} \mid d(x, A^{\mathcal{I}}) < d(x, B^{\mathcal{I}})\}.$$

$\mathcal{CSL}$  is a logic of pure qualitative comparisons, this motivates an alternative semantics where the distance function is replaced by preferential relations. In [2] it is shown that in non-symmetric minspaces the distance function can be replaced by a family of ternary relations. Although the same encoding can be also used for symmetric minspaces, we use here a binary relation on two-element multisets, which is more natural in this case and eases the formulation of both the semantics and the tableau calculus.

Given a non-empty set  $\Delta$ , we denote by  $\mathcal{MS}_2(\Delta)$  the set of two-element multisets over  $\Delta$ ; its elements are denoted by  $\{a, b\}$  (abusing set-notation). Intuitively, each  $\{a, b\}$  represents the distance from  $a$  to  $b$ . We define for all  $C \subseteq \mathcal{MS}_2(\Delta)$ ,  $x \in \Delta$  and  $A \subseteq \Delta$ :

$$\begin{aligned} \min_{<}(C) &= \{\{x, y\} \mid \{x, y\} \in C \text{ and } \forall \{u, v\} \in C, \{u, v\} \not< \{x, y\}\}, \\ \min_x(A) &= \{a \mid \{x, a\} \in \min(\{\{x, a\} \mid a \in A\})\}. \end{aligned}$$

As the relation  $<$  represents distance comparisons, it is expected to satisfy some specific properties:

<sup>2</sup> The case of the well known triangular inequality, which is not assumed here, is discussed in [8].

<sup>3</sup> If one of those sets is empty, the distance is by convention infinite.

**Definition 2.** Let  $\Delta$  be a non empty set. We say that a relation  $<$  defined on  $\mathcal{MS}_2(\Delta)$  satisfies:

- pair-centering if  $\forall x, y \in \Delta: \{x, x\} < \{x, y\}$  or  $x = y$ .
- modularity if  $\forall x, y, z, u \in \Delta$ :  
 $\{x, y\} < \{z, u\}$  implies  $\forall v, w \in \Delta, \{x, y\} < \{v, w\}$  or  $\{v, w\} < \{z, u\}$ .
- asymmetry if  $\forall x, y, z, u \in \Delta: \{x, y\} \not< \{z, u\}$  or  $\{z, u\} \not< \{x, y\}$ .
- limit assumption if  $\forall C, D \subseteq \Delta: C \neq \emptyset$  and  $D \neq \emptyset$  implies  
 $\min_{<} \{\{x, y\} \mid x \in C, y \in D\} \neq \emptyset$ .

In a few words, pair-centering corresponds to the property (id) of the distance function, asymmetry to the fact that the relation  $<$  encode strict comparisons, the limit assumption to the minspace property (min), whereas modularity is related to the fact that distance values are linearly ordered. We can now restate the definition of a  $\mathcal{CSL}$ -model in terms of preferential structures of this kind:

**Definition 3 (Pair Model).** A  $\mathcal{CSL}$ -pair-model  $\mathcal{I}$  is a triple  $\langle \Delta, <, \cdot^{\mathcal{I}} \rangle$  where:

- $\Delta$  is a non empty set.
- $<$  is a binary relation on  $\mathcal{MS}_2(\Delta)$  satisfying asymmetry, pair-centering, modularity and limit assumption.
- $\cdot^{\mathcal{I}}$  is the evaluation function defined as usual on propositional variables and boolean connectives, and as follow for the concept similarity operator:  
 $(A \Leftarrow B)^{\mathcal{I}} = \{x \in \Delta \mid \exists y \in A^{\mathcal{I}}, \forall z \in B^{\mathcal{I}}, \{x, y\} < \{x, z\}\}$ .

We can show that the pair semantics is equivalent to the distance semantics. We say that two models  $\mathcal{I}$  and  $\mathcal{J}$  are *equivalent* if they are based on the same set  $\Delta$  and for all formula  $A \in \mathcal{L}_{\mathcal{CSL}}$ ,  $A^{\mathcal{I}} = A^{\mathcal{J}}$ .

**Theorem 4.** For every pair model there is an equivalent symmetric minspace model, and vice versa.

### 3 Filtration

In this section we prove a filtration theorem for pair semantics of  $\mathcal{CSL}$ . As usual, for every formula  $C$ , let  $\text{sub}(C)$  denote a set of all subformulas of  $C$ . Clearly, the size of  $\text{sub}(C)$  does not exceed the length of  $C$  in symbols.

Let  $C$  be a fixed formula and  $\mathcal{I}$  be a pair model satisfying the limit assumption. A *type* of an element  $x$  in a pair model  $\mathcal{I}$  (with respect to  $C$ ), denoted by  $\tau^C(x)$ , is the set of all formulas in  $\text{sub}(C)$  whose interpretations in  $\mathcal{I}$  contain  $x$ , that is,  $\tau^C(x) \stackrel{\text{def}}{=} \{D \in \text{sub}(C) \mid x \in D^{\mathcal{I}}\}$ . Let  $n$  be the length of  $C$  in symbols. Since every type is a subset of  $\text{sub}(C)$ , the number of different types with respect to  $\text{sub}(C)$  does not exceed  $2^n$ .

Let  $\sim$  be an equivalence on  $\Delta^{\mathcal{I}}$  such that  $x \sim y$  implies  $\tau^C(x) = \tau^C(y)$  for every  $x$  and  $y$  from  $\Delta^{\mathcal{I}}$ . Based on a pair model  $\mathcal{I}$ , we will define a filtrated pair model  $\bar{\mathcal{I}}$  as follows. For every element  $x$  of  $\Delta^{\mathcal{I}}$ , let  $[x]$  be the equivalence class (with respect to  $\sim$ ) of the representative  $x$ , that is  $[x] \stackrel{\text{def}}{=} \{y \in \Delta^{\mathcal{I}} \mid x \sim y\}$ . The domain of  $\bar{\mathcal{I}}$  is defined as a set of all equivalence classes in  $\Delta^{\mathcal{I}}$  (with respect to

$\sim$ ), i.e.  $\Delta^{\bar{\mathcal{I}}} \stackrel{\text{def}}{=} \{[x] \mid x \in \Delta^{\mathcal{I}}\}$ . The relation  $<^{\bar{\mathcal{I}}}$  on  $\mathcal{MS}_2(\Delta^{\bar{\mathcal{I}}})$  is defined as follows. For all  $[x_0], [y_0], [x_1], [y_1]$  from  $\Delta^{\bar{\mathcal{I}}}$  we let

$$\{[x_0], [y_0]\} <^{\bar{\mathcal{I}}} \{[x_1], [y_1]\} \stackrel{\text{def}}{\iff} \exists x'_0 \in [x_0] \exists y'_0 \in [y_0] \forall x'_1 \in [x_1] \forall y'_1 \in [y_1] \{x'_0, y'_0\} <^{\mathcal{I}} \{x'_1, y'_1\}.$$

Finally, for every propositional variable  $P$  we define  $P^{\bar{\mathcal{I}}} \stackrel{\text{def}}{=} \{[x] \mid x \in P^{\mathcal{I}}\}$ . It is not difficult to see that the definition of  $\bar{\mathcal{I}}$  does not depend on the choice of representatives of the equivalence classes, so that  $\bar{\mathcal{I}}$  is defined correctly.

**Lemma 5.**  $\bar{\mathcal{I}}$  is a  $\mathcal{CSL}$  pair model.

**Lemma 6 (Filtration Lemma).** Let  $\mathcal{I}$  be a pair min-model.

Then, for every formula  $D \in \text{sub}(C)$  and  $x \in \Delta^{\mathcal{I}}$ ,  $x \in D^{\mathcal{I}} \iff [x] \in D^{\bar{\mathcal{I}}}$ .

As a consequence, we obtain the following theorem.

**Theorem 7 (Effective Finite Model Property).** A formula of a length  $n$  is satisfiable in a  $\mathcal{CSL}$  pair model iff it is satisfiable in a finite  $\mathcal{CSL}$  pair model where the number of elements does not exceed  $2^n$ .

## 4 A Tableau calculus

In this section we present a decision procedure for  $\mathcal{CSL}$  over symmetric minspaces based on a labeled tableau calculi. Tableau rules act on sets of tableau formulas, defined next. These sets of formulas are denoted by  $\Gamma, \Delta, \dots$ , and are called tableau sets. As usual, we use the notation  $\Gamma, \Delta$  for the set union  $\Gamma \cup \Delta$ . Given an enumerable set  $\text{Lab} = \{x_1, x_2, \dots, x_n, \dots\}$  of objects called *labels*, a tableau formula has the form i)  $x : C$  where  $x$  is a label and  $C$  is a  $\mathcal{CSL}$ -formula. ii)  $\{x, y\} < \{z, u\}$  or  $\{x, y\} \not< \{z, u\}$  where  $\{x, y\}, \{z, u\}$  denote two-element multisets of labels. iii)  $x : f(y, C)$  where  $x, y$  are labels and  $C$  is a  $\mathcal{CSL}$ -formula.

As expected, labels represent the objects of the domain, so that i) states that  $x \in C^{\mathcal{I}}$ , and ii) encode the preferential relation. Note that the calculus makes use of the dual relation  $\not<$  (which can be seen as a non-strict relation  $\geq$ ). The intuitive meaning of iii) is  $x \in \min_y(C^{\mathcal{I}})$ .

A tableau derivation (or simply a derivation) for  $C$  (the input formula) is a tree whose root node is the tableau set  $\{x : C\}$ , and where successors of any node are obtained by the application of a *tableau rule*. A tableau rule has the form  $\Gamma[X]/\Gamma_1 \mid \dots \mid \Gamma_m$ , where  $\Gamma, \Gamma_1, \dots, \Gamma_m$  are tableau sets. The meaning is that, given a tableau set  $\Gamma$  in a derivation, if  $X \subseteq \Gamma$ , then we can apply the rule to it and create  $m$  successors  $\Gamma_1, \dots, \Gamma_m$ . The denominator can be the empty set, in which case the rule is a *closure rule* and is usually written  $\Gamma[X]/\perp$ . Closure rules detect tableau sets which contain a contradiction.

The rules for our calculus are given in Figure 1. The rules are grounded on the pair-model semantics. Observe that the rules for  $\Leftarrow$  comprise a case analysis in the form of analytic cut. The semantic conditions for  $(\neg) \Leftarrow$  are captured by

$$\begin{array}{l}
(\neg\neg): \frac{\Gamma[x : \neg\neg A]}{\Gamma, x : A} \\
(\sqcup): \frac{\Gamma[x : A \sqcup B]}{\Gamma, x : A \mid \Gamma, x : B} \\
(\sqcap): \frac{\Gamma[x : A \sqcap B]}{\Gamma, x : A, x : B} \\
\\
(\text{T1}\Leftarrow): \frac{\Gamma[x : A \Leftarrow B, y : C]}{\Gamma, y : \neg B \mid \Gamma, y : B} \\
(\text{T2}\Leftarrow): \frac{\Gamma[x : A \Leftarrow B]}{\Gamma, z : f(x, A), z : A} (*) \\
(\text{T3}\Leftarrow): \frac{\Gamma[x : A \Leftarrow B, z : f(x, A), y : B]}{\Gamma, \{x, z\} < \{x, y\}} \\
(\text{cnt}): \frac{\Gamma[x : A, y : B]}{\Gamma(y/x) \mid \Gamma, \{x, x\} < \{x, y\}, \{y, y\} < \{y, x\}} \\
(\text{mod}): \frac{\Gamma[\{x, y\} \not< \{v, w\}, \{v, w\} \not< \{t, u\}]}{\Gamma, \{x, y\} \not< \{t, u\}} \\
(\text{r}\perp): \frac{\Gamma[\{x, y\} < \{z, u\}, \{x, y\} \not< \{z, u\}]}{\perp} \\
\\
(\neg\sqcup): \frac{\Gamma[x : \neg(A \sqcup B)]}{\Gamma, x : \neg A, x : \neg B} \\
(\neg\sqcap): \frac{\Gamma[x : \neg(A \sqcap B)]}{\Gamma, x : \neg A \mid \Gamma, x : \neg B} \\
\\
(\text{F1}\Leftarrow): \frac{\Gamma[x : \neg(A \Leftarrow B), y : C]}{\Gamma, y : A \mid \Gamma, y : \neg A} \\
(\text{F2}\Leftarrow): \frac{\Gamma[x : \neg(A \Leftarrow B), y : A]}{\Gamma, z : f(x, B), z : B} (*) \\
(\text{F3}\Leftarrow): \frac{\Gamma[x : \neg(A \Leftarrow B), z : f(x, B), y : A]}{\Gamma, \{x, y\} \not< \{x, z\}} \\
(\text{asm}): \frac{\Gamma[\{x, y\} < \{z, u\}]}{\Gamma, \{z, u\} \not< \{x, y\}} \\
(\text{tr}): \frac{\Gamma[\{x, y\} < \{v, w\}, \{v, w\} < \{t, u\}]}{\Gamma, \{x, y\} < \{t, u\}} \\
(\perp): \frac{\Gamma[x : A, x : \neg A]}{\perp}
\end{array}$$

(\*)  $z$  is a new label not occurring in the current branch.

**Figure 1.** Tableau calculus  $T_{CS\mathcal{L}}$  for  $CS\mathcal{L}$ .

the rules (T2 $\Leftarrow$ ) and (T3 $\Leftarrow$ ) (resp. (F2 $\Leftarrow$ ) and (F3 $\Leftarrow$ )). For instance the rule (F2 $\Leftarrow$ ) introduces an element  $z \in \min_y(B^I)$ , whenever  $A$  is non-empty and the rule (F3 $\Leftarrow$ ) states that no  $A$ -element  $y$  can be closer to  $x$  than  $z$ . In the (cnt) rule, we denote by  $\Gamma(y/x)$  the tableau set obtained by replacing  $y$  by  $x$  in every tableau formula of  $\Gamma$ .

A rule is *dynamic* if it introduce a new label, and *static* if it does not. In this calculus, the rules (T2 $\Leftarrow$ ) and (F2 $\Leftarrow$ ) are dynamic, all the others are static. We can note that, apart from the centering rule, no rule deletes any formula from tableau sets. The centering rule is special due to the substitution part. The first denominator of the rule identifies two labels. Thus it does not really suppress formulas. An alternative to the use of substitutions would be to make use of equality between labels, but this would lead to a more complex calculus.

This calculus does not provide a decision procedure, since the interplay between dynamic rules (introducing new labels) and static rules (adding new formulas to which the dynamic rules could be applied) can lead to infinite derivations. In order to make our calculus terminating, we introduce in Definition 8 some restrictions on the application of the rules.

Given a derivation branch  $\mathbf{B}$  and two labels  $x$  and  $y$  occurring in it, we say that  $x$  is older than  $y$  if  $x$  has been introduced before  $y$  in the branch<sup>4</sup>. Note that this older relation is *well founded*. We also define  $\text{Lab}_\Gamma$  as the set of all labels occurring in a tableau set  $\Gamma$ , and  $\Pi_\Gamma(x) = \{A \mid A \in \mathcal{L}_{CS\mathcal{L}} \text{ and } x : A \in \Gamma\}$ .

**Definition 8 (Termination restrictions).**

<sup>4</sup> From a practical point of view, the order of introduction of the labels can be stored locally within a tableau set and does not require to inspect a whole derivation branch.

- Irredundancy restriction** 1. Do not apply a static rule  $\Gamma/\Gamma_1 \mid \dots \mid \Gamma_n$  to a tableau set  $\Gamma$  if for some  $1 \leq i \leq n$ ,  $\Gamma_i = \Gamma$ .
2. Do not apply the rule (T2 $\Leftarrow$ ) to some formula  $x : A \Leftarrow B$  in  $\Gamma$  if there exists some label  $u$  such that  $u : f(x, A)$  and  $u : A$  are in  $\Gamma$ .
3. Do not apply the rule (F2 $\Leftarrow$ ) to some formulas  $x : \neg(A \Leftarrow B), y : C$  if there exists some label  $u$  such that  $u : f(x, B)$  and  $u : B$  are in  $\Gamma$ .
- Subset blocking** Do not apply the rule (T2 $\Leftarrow$ ) to a formula  $x : A \Leftarrow B$ , or the rule (F2 $\Leftarrow$ ) to some formulas  $x : \neg(A \Leftarrow B), y : C$  if there exists some label  $z$  older than  $x$  and such that  $\Pi_\Gamma(x) \subseteq \Pi_\Gamma(z)$ .
- Centering restriction** Do not replace label  $y$  by  $x$  in an application of (cnt) to  $x : A, y : B$  if  $y$  is older than  $x$ .

The purpose of the termination restrictions is to prevent unnecessary applications of the rules that could lead to infinite derivations, as we prove in the next section. The subset blocking condition is similar to dynamic blocking [3]. The centering rule in itself is similar to the unrestricted blocking rule [7].

A rule  $R$  is *applicable* to a tableau set  $\Gamma$  under termination restrictions if it respects all termination restrictions (Definition 8). A derivation is *under termination restrictions* if all rule applications respects the termination restrictions. From now on, we only consider derivations under termination restrictions.

Since termination restrictions prevent the application of some rules we have to define whenever a tableau is open or closed, thereby witnessing the satisfiability (or unsatisfiability) of the input formula.

**Definition 9 (Finished / Bad / Good tableau sets, Finished / Closed / Open derivation).** A tableau set is *finished* if it is closed or no rule is applicable to it.

A tableau set  $\Gamma$  in a derivation is *bad* if: (i) it is open. (ii) it is finished. (iii) there exists some labels  $x$  and  $y$  such that  $x$  is older than  $y$  and  $\Pi_\Gamma(y) \subseteq \Pi_\Gamma(x)$ .

A tableau set is *good* if it is open, finished, and not bad.

A derivation is *finished* if all its leaf nodes are finished, it is *closed* if all its leaf nodes are either closed or bad, and it is *open* if it contains a good tableau set.

Here is an intuitive account of the above definitions. Finished tableau sets are leaf nodes in the derivation tree. Closed tableau sets contain a contradiction and thus they cannot provide a model for the input formula. Good tableau sets represent (partial) models of the input formula. Bad tableau sets are disregarded as potential models, for the reason explained below. An open tableau derivation provide a model of the input formula, whereas a closed tableau derivation shows that the input formula is unsatisfiable.

The status of a bad tableau set is unknown: it does not contain a contradiction, but we do not know whether it can provide a model or not. In any case, we disregard it, as it may potentially provide an infinite model which would violate the limit assumption property. To this regard, observe that the limit assumption property is not encoded by any rule of the calculus. It is the whole tableau construction that takes care to eliminate potential infinite models violating this condition, in accordance with the finite model property shown in the previous

section. Disregarding bad tableau sets is essential in order to obtain a complete calculus, as the example below shows. Of course we have to prove that ignoring *bad* tableau sets preserves the soundness of the calculus.

*Example 10.* We consider the example in the introduction of a 'cyclic' KB about cars. Observe first that we can encode an inclusion  $P \sqsubseteq Q$  by  $\neg(\neg(\neg P \sqcup Q) \Leftarrow \perp)$ , for which we get the derived rule  $\frac{\Gamma[x:\neg(\neg(\neg P \sqcup Q) \Leftarrow \perp), y:R]}{\Gamma, y:\neg P \sqcup Q}$ . Suppose now we initialise then the tableau by

$$\Gamma = \{x_1 : G \sqsubseteq (F \Leftarrow C), x_1 : F \sqsubseteq (C \Leftarrow G), x_1 : C \sqsubseteq (G \Leftarrow F), x_1 : C\}$$

Due to space limitations, we only show the generation of the tableau set built according to the termination restrictions that cannot be closed by simple rule applications. We apply obvious propositional simplifications (e.g. if a tableau set contains both  $A$  and  $\neg A \sqcup B$ , we only add  $B$ ). A first set of applications of the rules add to  $\Gamma$ :

$$x_1 : G \Leftarrow F, x_1 : \neg F \sqcup (C \Leftarrow G), x_1 : \neg G \sqcup (F \Leftarrow C), x_1 : \neg F, x_1 : \neg G$$

To see the last two: by applying  $(T1 \Leftarrow)$  to  $x_1 : G \Leftarrow F$ , the branch with  $x_1 : F$  gives a closed tableau set; similarly, expanding  $\neg G \sqcup (F \Leftarrow C)$ , the tableau set with  $x_1 : F \Leftarrow C$  will be closed because of  $x_1 : C$ . In the following we tacitly apply a similar pruning of the tableau derivation. We now apply  $(T2 \Leftarrow)$  to  $x_1 : G \Leftarrow F$  and then the same static rules as before, so that we add:

$$z_1 : f(x_1, G), z_1 : G, z_1 : F \Leftarrow C, z_1 : \neg F \sqcup (C \Leftarrow G), z_1 : \neg C \sqcup (G \Leftarrow F), z_1 : \neg F, z_1 : \neg C.$$

We continue by applying  $(T2 \Leftarrow)$  to  $z_1 : F \Leftarrow C$  we add:

$$u_1 : f(z_1, F), u_1 : F, u_1 : C \Leftarrow G, u_1 : \neg C \sqcup (G \Leftarrow F), u_1 : \neg C \sqcup (G \Leftarrow F), u_1 : \neg C, u_1 : \neg G.$$

We continue by applying  $(T2 \Leftarrow)$  to  $u_1 : C \Leftarrow G$  so that we add:

$$x_2 : f(u_1, C), x_2 : C, x_2 : G \Leftarrow F, x_2 : \neg F \sqcup (C \Leftarrow G), x_2 : \neg G \sqcup (F \Leftarrow C), x_2 : \neg F, x_2 : \neg G$$

Now we apply  $(T3 \Leftarrow)$  and then (tr) we add:

$$\{x_1, z_1\} < \{x_1, u_1\}, \{z_1, u_1\} < \{z_1, x_2\}, \{z_1, u_1\} < \{z_1, x_1\}, \{u_1, x_2\} < \{u_1, z_1\}, \\ \{z_1, u_1\} < \{x_1, u_1\}, \{u_1, x_2\} < \{z_1, x_1\}, \{u_1, x_2\} < \{u_1, x_1\}$$

The last three are obtained by (tr). By (asm) and (mod) we add:

$$\{x_1, u_1\} \not< \{x_1, z_1\}, \{z_1, x_2\} \not< \{z_1, u_1\}, \{z_1, x_1\} \not< \{z_1, u_1\}, \{u_1, z_1\} \not< \{u_1, x_2\}, \\ \{x_1, u_1\} \not< \{z_1, u_1\}, \{z_1, x_1\} \not< \{u_1, x_2\}, \{u_1, x_1\} \not< \{u_1, x_2\}$$

We now apply (cnt): for every pair of labels, except  $x_1$  and  $x_2$ , the left tableau set involving substitution will be closed, so that relations  $\{x_1, x_1\} < \{x_1, u_1\}$ ,  $\{z_1, z_1\} < \{x_1, u_1\} \dots$  will be added. For  $x_1$  and  $x_2$ , we get two tableau set  $\Gamma_1 = \Gamma[x_2/x_1]$  and  $\Gamma_2 = \Gamma, \{x_1, x_1\} < \{x_1, x_2\}, \{x_2, x_2\} < \{x_1, x_2\}$ . The former  $\Gamma_1$  is closed by (asm) and  $(r\perp)$  since it contains  $\{u_1, x_1\} < \{u_1, x_1\}$ . The latter  $\Gamma_2$  is finished (and open): in particular we cannot apply  $(T2 \Leftarrow)$  to  $x_2 : G \Leftarrow F$  by the fact that  $\Pi_{\Gamma_2}(x_2) \subseteq \Pi_{\Gamma_2}(x_1)$  and by the subset blocking restriction. Thus  $\Gamma_2$  is a *bad* tableau set and consequently it is disregarded. We can conclude that the tableau for the input formula(s) is *closed*. Had we not applied subset blocking restriction, the construction would have produced an infinite tableau set with an infinite sequence of "closer" and "closer" pairs of labels (wrt. to

<). To this regard, observe that the initial formula is *satisfiable* in a symmetric infinite model *which is not a minspace*.

## 5 Main Results

In this section we prove termination, soundness and completeness of the calculus.

### 5.1 Termination

**Theorem 11.** *Any derivation of  $T_{CS\mathcal{L}}$  under termination restrictions is finite.*

To prove the theorem we need some additional definitions. We define, for any tableau derivation  $T$ ,  $\Pi(T)$  as the set of all  $CS\mathcal{L}$ -formulas occurring in  $T$ :

$$\Pi(T) \stackrel{\text{def}}{=} \{A \mid A \in \mathcal{L}_{CS\mathcal{L}} \text{ and } x : A \in \Gamma \text{ for some } \Gamma \text{ from } T\}.$$

Given a node  $\Gamma$  in a branch  $\mathbf{B}$ , and  $x$  occurring in  $\Gamma$  or in any of its ancestors, we also define  $\sigma^*(\Gamma, x) = x$  if  $x$  occurs in  $\Gamma$ , or  $\sigma^*(\Gamma, x) = y$  where  $y$  is the label which finally replaces  $x$  in  $\Gamma$  along a sequence of centering substitution applied on the path between  $\Gamma$  and its closest ancestor in which  $x$  occurs. Observe that this label  $y$  is unique, and is older than  $x$  (due to the centering restriction).

**Proposition 12 (Monotonicity).** *Let  $\Gamma$  be a tableau set in a derivation  $T$ , and  $\Gamma'$  be any descendant of  $\Gamma$  in  $T$ . Let  $x : A$  be in  $\Gamma$ . Then: (i)  $\Pi_\Gamma(x) \subseteq \Pi_{\Gamma'}(\sigma^*(\Gamma', x))$ . (ii) if  $x : A$  is blocked for a rule  $R$  in  $\Gamma$  by the irredundancy restriction, then also  $\sigma^*(\Gamma', x) : A$  is blocked in  $\Gamma'$  for the rule  $R$  by the irredundancy restriction.*

*Proof (Theorem 11).* Let  $T$  be a derivation for  $C$ . By absurd, suppose that  $T$  contains an infinite branch  $\mathbf{B} = (\Gamma_i)_{i \in \mathbb{N}}$ . We can then prove the following:

**FACT 13.** There exists a dynamic formula  $D = (\neg)(A \not\equiv B)$ , a subsequence  $(\Theta_n)_{n \in \mathbb{N}}$  of  $\mathbf{B}$ , and a sequence  $(x_n)_{n \in \mathbb{N}}$  of labels such that: (i) For all  $n$ ,  $x_n : D \in \Gamma_n$ . (ii) For all  $n$ ,  $\Theta_n$  is a node corresponding to the application of the corresponding dynamic rule to the formula  $x_n : D$ . (iii) There exists a label  $x^*$  such that  $x^*$  occurs in each  $\Gamma_n$  and for all  $n$ ,  $\Pi_{\Theta_n}(x_n) \subseteq \Pi_{\Theta_n}(x^*)$

By the previous Facts, we have that (i) for all  $n$ ,  $x_n$  must be older than  $x^*$ , as if it were younger no dynamic rule could be applied to  $x_n : D$ , due to the subset blocking. (ii) for all  $n, m$ , if  $n \neq m$  then  $x_n \neq x_m$ , because the irredundancy restriction prevents multiple application of the same rule to the same formulas, due to Proposition 12. Therefore we have an infinite sequence of labels, all different and older than  $x^*$ , which is impossible as the relation older is well founded.  $\square$

We can estimate an upper bound of complexity of the tableau algorithm.

**Proposition 14.** *Let  $T$  be a tableau derivation for a formula  $C$  and  $n$  be the length of  $C$  in symbols. Then  $\text{Card}\{\Pi_\Gamma(x) \mid \Gamma \text{ in } T \text{ and } x \text{ occurs in } \Gamma\} \leq 2^{2n-1}$ .*

By Proposition 14, the derivation  $T$  cannot produce more than  $2^{2n-1}$  sets  $\Pi_\Gamma(x)$  for different tableau sets  $\Gamma$  and labels  $x$ . Due to the subset blocking condition in Definition 8, we cannot have more than  $2^{2n-1}$  labels in any branch of  $T$ . The number of  $\mathcal{CSL}$  formulas associated with any label is less than  $2n - 1$ , also there are  $2^{2n-1} \cdot (2n - 1)$  formulas of the kind  $f(x, A)$  and furthermore any tableau set contains less than  $2^{2n+1}$  preferential relations. Since the maximal number of premises in the tableau rules is 3 there can be no more than  $3 \cdot (2^{2n-1} \cdot (2n - 1) + 2^{2n} \cdot (2n - 1) + 2^{2n+1}) = 3 \cdot 2^{2n-1} \cdot (6n + 1)$  rule application steps in any branch of the derivation.

Thus, the number of steps in any branch of a tableau derivation is at most exponential in the length of an input formula. This shows that the non-deterministic tableau algorithm runs in NEXPTIME.

## 5.2 Soundness

We first need the following definitions:

**Definition 15 (CSL-mapping).** Let  $\Gamma$  be a tableau set and  $\mathcal{I}$  be a CSL-pair model. A CSL-mapping  $m$  from  $\Gamma$  to  $\mathcal{I}$  is a function  $m : \text{Lab}_\Gamma \rightarrow \Delta^\mathcal{I}$  such that:

- if  $\{x, y\} < \{z, u\} \in \Gamma$ , then  $\{m(x), m(y)\} <^\mathcal{I} \{m(z), m(u)\}$ .
- if  $\{x, y\} \not< \{z, u\} \in \Gamma$ , then  $\{m(x), m(y)\} \not<^\mathcal{I} \{m(z), m(u)\}$ .

We say that  $m$  is a min-mapping if it also satisfies the following property<sup>5</sup>: if  $z : f(x, A) \in \Gamma$ , then  $m(z) \in \min_{m(x)}(A^\mathcal{I})$ .

**Definition 16 (Satisfiability).** A tableau set  $\Gamma$  is satisfiable in a CSL-pair model  $\mathcal{I}$  under a CSL-mapping  $m$  iff:

- if  $x : A \in \Gamma$ , then  $m(x) \in A^\mathcal{I}$ .

A tableau set  $\Gamma$  is satisfiable if it is satisfiable in some model  $\mathcal{I}$  under some mapping  $m$ .

We then prove that, independently from the termination restrictions, the rules are sound, in the sense that they preserve satisfiability under a min-mapping.

**Theorem 17 (Soundness of the rules).** Let  $\Gamma$  be a tableau set satisfiable under a min-mapping, and let  $\Gamma/\Gamma_1 \mid \dots \mid \Gamma_n$  be an instance of any rule. Then there is an  $i$  such that  $\Gamma_i$  is satisfiable under a min-mapping.

**Corollary 18.** If  $C$  is satisfiable, then any tableau for  $C$  under termination restrictions contains a finished tableau set satisfiable under a min-mapping.

We now show the soundness of the tableau calculus with the terminating conditions. The previous corollary does not suffice, as the satisfiable finished tableau set could be *bad* and consequently disregarded.

**Theorem 19 (Soundness of the calculus).** If  $C$  is satisfiable, then any finished derivation started by  $x : C$  is open (that is it contains a good tableau set).

<sup>5</sup> Note that any min-mapping is also a CSL-mapping.

To prove the theorem, we need the following definition:

**Definition 20.** A tableau set  $\Gamma$  in a derivation is leftmost with respect to centering if (i) it is finished. (ii) for any labels  $x, y$  occurring in  $\Gamma$  (with  $x$  older than  $y$ ) the following holds: if there exists an ancestor  $\Delta$  of  $\Gamma$  to which (cnt)-rule has been applied with labels  $x$  and  $y$ , then  $\Delta(y/x)$  is not satisfiable.

**Proposition 21.** Given a derivation, any leftmost tableau set satisfiable under a min-mapping is good.

*Proof (Proposition 21).* Let  $\Gamma$  be a leftmost finished tableau set in a branch  $\mathbf{B}$ , such that  $\Gamma$  is satisfiable in a model  $\mathcal{I}$  under a min-mapping  $m$ . Suppose by absurd that  $\Gamma$  is bad. Then we can find two labels  $x$  and  $y$  in  $\Gamma$  such that: i)  $x$  is older than  $y$ . ii)  $x$  is the oldest label such that  $\Pi_\Gamma(y) \subseteq \Pi_\Gamma(x)$ . Notice that, being the oldest,  $x$  cannot be blocked in  $\Gamma$  by subset blocking.

FACT 22. There exists an ancestor  $\Delta$  of  $\Gamma$  such that  $\Delta$  is the node corresponding to an application of centering to  $x$  and  $y$ . Moreover,  $\Delta$  is satisfiable in  $\mathcal{I}$  under some min-mapping  $m'$ .

We now build a model  $\mathcal{I}'$  by doing a “partial filtration” of  $\mathcal{I}$  in which we just identify the elements  $m(x)$  and  $m(y)$ . To this purpose, we first define the relation  $\sim_{x,y}$  as follows: i)  $\forall u \in \Delta^{\mathcal{I}}, u \sim_{x,y} u$ . ii)  $m(x) \sim_{x,y} m(y)$  and  $m(y) \sim_{x,y} m(x)$ . It is easy to see that  $\sim_{x,y}$  is an equivalence relation. For all  $u \in \Delta^{\mathcal{I}}$ , we note its equivalence class by  $[u]$ . We now build  $\mathcal{I}'$  as follows:

- $\Delta^{\mathcal{I}'} = \{[u] \mid u \in \Delta^{\mathcal{I}}\}$ .
- For all  $[u], [v], [w], [z]$ , we let  $\{[u], [v]\} <^{\mathcal{I}'} \{[w], [z]\}$  iff  $\exists u' \in [u], \exists v' \in [v]$ , such that  $\forall w' \in [w], \forall z' \in [z], \{u', v'\} <^{\mathcal{I}} \{w', z'\}$ .
- For all propositional variables  $P$  and for all  $u \neq m(y)$ ,  $[u] \in P^{\mathcal{I}'}$  iff  $u \in P^{\mathcal{I}}$ . Since  $[m(y)] = [m(x)]$ , for all propositional variables  $P$ ,  $[m(y)] \in P^{\mathcal{I}'}$  iff  $m(x) \in P^{\mathcal{I}}$ .

FACT 23. 1.  $\mathcal{I}'$  is a  $\mathcal{CSL}$ -pair model. 2.  $m'' : u \rightarrow [m'(u)]$  is min-mapping from  $\Delta(y/x)$  to  $\mathcal{I}'$ . 3.  $\Delta(y/x)$  is satisfiable in  $\mathcal{I}'$  under  $m''$ .

By Fact 23, we conclude that  $\Gamma$  is not leftmost, which leads to the contradiction.  $\square$

We are now ready to prove the Theorem 19.

*Proof (Theorem 19).* Let  $C$  be a satisfiable  $\mathcal{CSL}$ -formula and let  $T$  be a derivation for  $C$ . By corollary 18,  $T$  contains a finished tableau set satisfiable under some min-mapping. Since  $T$  is finite, it also contains a leftmost tableau set which is satisfiable under some min-mapping. By Proposition 21, this tableau set is good, thus  $T$  is open.  $\square$

### 5.3 Completeness

**Theorem 24.** If  $\Gamma$  is a good tableau set, then  $\Gamma$  is satisfiable.

As usual, the completeness is proved by building a canonical model  $\mathcal{I}_\Gamma$  from the good tableau set  $\Gamma$ . Please note that if  $\Gamma$  is a good tableau set,  $\Gamma$  is finite and contains no labels blocked by the subset blocking of Definition 8. Consequently  $\Gamma$  is saturated, where the notion of saturation is the usual one (closure under the rules). The main issue for building the canonical model is that the relation  $<$  in  $\Gamma$  is not modular. We have to modularize it in order to define the model.

The relation  $\not\prec$  in  $\Gamma$  can be seen as a partial relation  $\geq$ . From it we can define a binary relation  $=_{\not\prec}$  as follow:

- for all  $x, y \in \text{Lab}_\Gamma$ ,  $\{x, y\} =_{\not\prec} \{x, y\}$ .
- for all  $x, y, u, z \in \text{Lab}_\Gamma$ , if  $\{x, y\} \not\prec \{u, v\}$  and  $\{u, v\} \not\prec \{x, y\}$  are in  $\Gamma$ , then  $\{x, y\} =_{\not\prec} \{u, v\}$ .

It is easy to see that  $=_{\not\prec}$  is an equivalence relation (transitivity comes from the rule (mod)). We denote by  $[\{x, y\}]_{\not\prec}$  the equivalence class of  $\{x, y\}$  modulo the relation  $=_{\not\prec}$ . Intuitively, if we see  $\{x, y\}$  as the distance from  $x$  to  $y$ , then  $\{x, y\} =_{\not\prec} \{u, v\}$  means that  $d(x, y) = d(u, v)$ .

The relation  $\not\prec$  in  $\Gamma$  also induce a relation  $<_e$  between the equivalence classes: we say that  $[\{x, y\}]_{\not\prec} <_e [\{u, v\}]_{\not\prec}$  iff there exists  $\{x', y'\} \in [\{x, y\}]_{\not\prec}$  and  $\{u', v'\} \in [\{u, v\}]_{\not\prec}$  such that  $\{u', v'\} \not\prec \{x', y'\}$  is in  $\Gamma$  and  $\{x', y'\} \not\prec \{u', v'\}$  is *not* in  $\Gamma$ .

FACT 25. The relation  $<_e$  over the equivalence classes is a (strict) partial order.

We let  $<^*$  be any linear extension of  $<_e$  (by doing for instance a topological sort on  $<_e$ ). We can now build the canonical model  $\mathcal{I}_\Gamma = \langle \Delta^{\mathcal{I}_\Gamma}, <^{\mathcal{I}_\Gamma}, \cdot^{\mathcal{I}_\Gamma} \rangle$  as follow:

- $\Delta^{\mathcal{I}_\Gamma} = \text{Lab}_\Gamma$ .
- For all  $x, y, u, v \in \Delta^{\mathcal{I}_\Gamma}$ ,  $\{x, y\} <^{\mathcal{I}_\Gamma} \{u, v\}$  iff  $[\{x, y\}]_{\not\prec} <^* [\{u, v\}]_{\not\prec}$ .
- For all propositional variable  $P$ :  $P^{\mathcal{I}_\Gamma} = \{x \mid x : P \in \Gamma\}$ .

FACT 26.  $\mathcal{I}_\Gamma$  is a  $\mathcal{CSL}$ -model, and  $\Gamma$  is satisfiable in it under the identity mapping  $\text{id} : x \rightarrow x$ .

## 6 Non-symmetric case

With minimal changes, we can also handle the non-symmetric case. In this case, the distance function of minspaces is not assumed to be symmetric. Minspace distance models are defined exactly as in section 2 and include obviously symmetric minspace models as a proper subclass. We still have a correspondence between non-symmetric minspace models, that is arbitrary minspace models, and pair preferential models where the preference relation is defined over ordered pairs  $(x, y) < (x, z)$  of elements having the same first component. Let us call them *non-symmetric pair models*. In these models the preference relation over pairs  $(x, y) < (x, z)$  can actually be replaced by a ternary relation  $y <_x z$ , as it is done for instance in [2].

Concerning the tableau calculus, the only thing to change is how we represent preferential relations: the two-element multisets  $\{x, y\}$  now become *ordered pairs*. Thus in this setting  $(x, y) \neq (y, x)$ . Using this notation, we can reformulate

$$\begin{array}{l}
(\text{T3}\equiv): \frac{\Gamma[x : A \equiv B, z : f(x, A), y : B]}{\Gamma, (x, z) < (x, y)} \\
(\text{cnt}): \frac{\Gamma[x : A, y : B]}{\Gamma(y/x) \mid \Gamma, (x, x) < (x, y), (y, y) < (y, x)} \\
(\text{mod}): \frac{\Gamma[(x, y) \not< (v, w), (v, w) \not< (t, u)]}{\Gamma, (x, y) \not< (t, u)} \\
(\text{r}\perp): \frac{\Gamma[(x, y) < (z, u), (x, y) \not< (z, u)]}{\perp} \\
(\text{F3}\equiv): \frac{\Gamma[x : \neg(A \equiv B), z : f(x, B), y : A]}{\Gamma, (x, y) \not< (x, z)} \\
(\text{asm}): \frac{\Gamma[(x, y) < (z, u)]}{\Gamma, (z, u) \not< (x, y)} \\
(\text{tr}): \frac{\Gamma[(x, y) < (v, w), (v, w) < (t, u)]}{\Gamma, (x, y) < (t, u)}
\end{array}$$

**Figure 2.** Rules of  $T_{\text{CSLNS}}$ .

the rules of  $T_{\text{CSL}}$  in figure 1 to obtain a calculus  $T_{\text{CSLNS}}$  for  $\text{CSL}$  over non-symmetric pair models, corresponding to non-symmetric (or arbitrary) minspace models. The new rules are presented in figure 2. Termination, soundness and completeness proofs still holds for this case, as they are independant of the encoding of the pairs.

**Theorem 27.** *The calculus  $T_{\text{CSLNS}}$  gives a terminating, sound and complete decision procedure for  $\text{CSL}$  over non-symmetric minspaces.*

## 7 Conclusions and related works

In this work we have contributed to the study of automated deduction for the logic of comparative concept similarity  $\text{CSL}$  interpreted over symmetric (and non-symmetric) minspaces. Our first result is a proof of the finite model property with respect to the preferential semantics by means of a finite filtration method. Then we have presented a labelled tableau calculus which gives a practically implementable decision procedure for this logic. Termination of the calculus is obtained by imposing suitable restrictions reminiscent of dynamic blocking and unrestricted blocking known in the literature [3,7]. Our calculi gives a NEXP-TIME decision procedure for deciding satisfiability of  $\text{CSL}$  formulas. However, it is not optimal in the light of the known EXPTIME upper bound.

The original paper [8] for  $\text{CSL}$  reduces  $\text{CSL}$  satisfiability to checking emptiness of an automaton built from an input concept. This automaton based algorithm for checking  $\text{CSL}$  satisfiability runs in EXPTIME. However, it requires a complex construction of a large automaton which turns in many simple cases to be excessive. On the contrary, tableau algorithms construct tableaux on-the-fly and terminate straight away when derived information suffices to determine the status of the input formula. Thus, despite that the complexity of the proposed tableau algorithm is higher than the original automaton based algorithm, we expect that our tableau algorithm will perform better on some large classes of problems. Moreover, known techniques (such as caching) could be employed to make the tableau calculus more efficient. In a future perspective, a practical comparison of the two algorithms will look very interesting.

In [2] a labelled tableau calculus for  $\text{CSL}$  over non symmetric minspaces is presented; a theorem prover implementing the calculus is described in [1]. This

calculus is similar to the one introduced in this work (for the non-symmetric case) as they are both based on the preferential semantics. However, it makes use of a family of modalities indexed on labels/objects and needs specific rules for handling them. The modal rules (similar to modal logic GL) force the limit-assumption property with respect to the ternary preference relation. Moreover termination is obtained by imposing relatively complex blocking restrictions. In contrast, the calculus presented here has only rules for  $\mathcal{CSL}$ -connectives, and it is expectedly more efficient. It is not obvious how to extend the approach of [2] to the symmetric case, in particular how to capture the limit assumption property by means of modal rules.

Finally the calculus may be extended to treat a wider language, more interesting for description-logic applications. For instance, we expect that we can easily treat individuals (nominals). Being a labelled tableau, handling individuals should not be too difficult. For this purpose the calculus should integrate equality reasoning on individuals and the left conclusion of the (cnt) rule should make use of explicit equality rather than substitution. However, the soundness proof may require some effort to adapt the involved partial filtration construction.

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[Appendix: Details of the proofs]

## A Syntax and Semantics

*Proof (Theorem 4).* (Pair Model  $\Rightarrow$  Symmetric Minspace model)

Let  $\mathcal{I} = \langle \Delta, <, \cdot^{\mathcal{I}} \rangle$  be a  $\mathcal{CSL}$ -pair model. As  $<$  is modular, there is a ranking function [5]  $r$  such that for all  $X, Y \in \mathcal{MS}_2(\Delta)$ :  $X < Y$  iff  $r(X) < r(Y)$ . We now define the distance  $d$  on  $\Delta$  by taking: (i)  $d(a, b) = r(\{a, b\})$  if  $a \neq b$ . (ii)  $d(a, b) = 0$  if  $a = b$ .  $d$  satisfies (id) and (sym) (as  $\{a, b\} = \{b, a\}$ ). Thus we can build a minspace model  $\mathcal{J} = \langle \Delta, d, \cdot^{\mathcal{J}} \rangle$  by taking  $P^{\mathcal{J}} = P^{\mathcal{I}}$  for every propositional variable  $P$ . It is then easy to check, by induction on the structure of the formulas, that  $\mathcal{J}$  is equivalent to  $\mathcal{I}$ .

(Symmetric Minspace model  $\Rightarrow$  Pair Model) Let  $\mathcal{J} = \langle \Delta, d, \cdot^{\mathcal{J}} \rangle$  be a  $\mathcal{CSL}$  minspace model. We then define  $\mathcal{I} = \langle \Delta, <, \cdot^{\mathcal{I}} \rangle$  as follow: (i)  $\{x, y\} < \{u, z\}$  if  $d(x, y) < d(u, z)$ . (ii)  $P^{\mathcal{I}} = P^{\mathcal{J}}$  for every propositional variable  $P$ . It is easy to check that  $<$  satisfies pair-centering, modularity, asymmetry and the limit assumption, and thus, by induction on the formula's structure, that  $\mathcal{I}$  is a  $\mathcal{CSL}$ -pair model equivalent to  $\mathcal{J}$ .  $\square$

## B Filtration

*Proof (Lemma 5).* It is enough to check that the conditions pair-centering, asymmetry, modularity and limit assumption are all satisfied for  $\bar{\mathcal{I}}$ .

**Pair-centering.** Let  $\{[x_0], [x_0]\} \not\prec^{\bar{\mathcal{I}}} \{[x_1], [y_1]\}$ . That is, for every  $x, y \in [x_0]$  there are  $x'_1 \in [x_1]$  and  $y'_1 \in [y_1]$  such that  $\{x, y\} \not\prec^{\mathcal{I}} \{x'_1, y'_1\}$ . In particular, there are  $x'_1 \in [x_1]$  and  $y'_1 \in [y_1]$  such that  $\{x_0, x_0\} \not\prec^{\mathcal{I}} \{x'_1, y'_1\}$ . The model  $\mathcal{I}$  satisfies the pair-centering condition and, hence,  $x'_1 = y'_1$ . Therefore,  $[x_1] = [y_1]$ .

**Modularity.** Let  $\{[x_0], [y_0]\} <^{\bar{\mathcal{I}}} \{[x_1], [y_1]\}$ . That is, there are  $x'_0 \in [x_0]$  and  $y'_0 \in [y_0]$  such that  $\{x'_0, y'_0\} <^{\mathcal{I}} \{x, y\}$  for all  $x \in [x_1]$  and  $y \in [y_1]$ . Assume that  $\{[x_0], [y_0]\} \not\prec^{\bar{\mathcal{I}}} \{[x_2], [y_2]\}$ . Thus, for all  $x \in [x_0]$  and  $y \in [y_0]$  there are  $x'_2 \in [x_2]$  and  $y'_2 \in [y_2]$  such that  $\{x, y\} \not\prec^{\mathcal{I}} \{x'_2, y'_2\}$ . In particular, there are  $x'_2 \in [x_2]$  and  $y'_2 \in [y_2]$  such that  $\{x'_0, y'_0\} \not\prec^{\mathcal{I}} \{x'_2, y'_2\}$ . Finally, by the modularity condition for  $\mathcal{I}$ , we have  $\{x'_2, y'_2\} <^{\mathcal{I}} \{x, y\}$  for all  $x \in [x_1]$  and  $y \in [y_1]$ .

**Asymmetry.** Let  $\{[x_0], [y_0]\} <^{\bar{\mathcal{I}}} \{[x_1], [y_1]\}$ . That is, there are  $x'_0 \in [x_0]$  and  $y'_0 \in [y_0]$  such that  $\{x'_0, y'_0\} <^{\mathcal{I}} \{x, y\}$  for all  $x \in [x_1]$  and  $y \in [y_1]$ . By the asymmetry condition for  $\mathcal{I}$ ,  $\{x, y\} \not\prec^{\mathcal{I}} \{x'_0, y'_0\}$  for all  $x \in [x_1]$  and  $y \in [y_1]$ . Thus, for all  $x \in [x_1]$  and  $y \in [y_1]$  there are  $x'_0 \in [x_0]$  and  $y'_0 \in [y_0]$  such that  $\{x, y\} \not\prec^{\mathcal{I}} \{x'_0, y'_0\}$ . That is,  $\{[x_1], [y_1]\} \not\prec^{\bar{\mathcal{I}}} \{[x_0], [y_0]\}$ .

**Limit assumption.** It is not difficult to see that any pair model satisfies the limit assumption if and only if it does not contain any infinitely decreasing sequence of pairs. Hence, the limit assumption is trivially satisfied in every finite pair model.

□

*Proof (Lemma 6).* We prove the statement by induction on the structure of  $D$ . The basis of induction and induction steps for Booleans are clear.

Let us prove the induction step for  $\Leftarrow$ . Let  $D = (E \Leftarrow F)$ . By the induction hypothesis  $E^{\bar{\mathcal{I}}} = \{[x] \mid x \in E^{\mathcal{I}}\}$  and  $F^{\bar{\mathcal{I}}} = \{[x] \mid x \in F^{\mathcal{I}}\}$ .

We consider three cases. In the first one we suppose that  $F^{\mathcal{I}} = \emptyset$  and  $E^{\mathcal{I}} = \emptyset$ . In this case we have  $F^{\bar{\mathcal{I}}} = \emptyset$  and  $E^{\bar{\mathcal{I}}} = \emptyset$  and, hence,  $D^{\mathcal{I}} = D^{\bar{\mathcal{I}}} = \emptyset$  and the lemma statement holds for  $D$  trivially.

In the second case, which is also trivial we suppose that  $F^{\mathcal{I}} = \emptyset$  and  $E^{\mathcal{I}} \neq \emptyset$ . Then  $F^{\bar{\mathcal{I}}} = \emptyset$  and  $E^{\bar{\mathcal{I}}} \neq \emptyset$  and, hence,  $D^{\mathcal{I}} = \Delta^{\mathcal{I}}$  and  $D^{\bar{\mathcal{I}}} = \Delta^{\bar{\mathcal{I}}}$  and again the lemma statement holds for  $D$ .

In the third and last case we assume that both  $F^{\mathcal{I}}$  and  $F^{\bar{\mathcal{I}}}$  are not empty. This is the non-trivial case and we proceed as follows. Let  $x \in (E \Leftarrow F)^{\mathcal{I}}$ . Then for all  $x' \in [x]$  and  $z \in F^{\mathcal{I}}$  there is  $y \in E^{\mathcal{I}}$  such that  $\{x', y\} <^{\mathcal{I}} \{x', z\}$ . Let  $[z] \in F^{\bar{\mathcal{I}}}$ . By the limit assumption we can find  $x_0 \in [x]$  and  $z_0 \in [z]$  such that  $\{x', z'\} \not<^{\mathcal{I}} \{x_0, z_0\}$  for all  $x' \in [x]$  and  $z' \in [z]$ . There is  $y \in E^{\mathcal{I}}$  such that  $\{x_0, y\} <^{\mathcal{I}} \{x_0, z_0\}$ . For all  $x' \in [x]$  and  $z' \in [z]$ , since  $\{x_0, y\} <^{\mathcal{I}} \{x_0, z_0\}$  and  $\{x', z'\} \not<^{\mathcal{I}} \{x_0, z_0\}$ , by the modularity condition, we have  $\{x_0, y\} <^{\mathcal{I}} \{x', z'\}$ . Thus,  $\{[x], [y]\} <^{\bar{\mathcal{I}}} \{[x], [z]\}$ .

Conversely, let  $[x] \in (E \Leftarrow F)^{\bar{\mathcal{I}}}$ . Let  $z \in F^{\mathcal{I}}$  be arbitrary. Then  $[z] \in F^{\bar{\mathcal{I}}}$  and, hence, there is  $[y] \in E^{\bar{\mathcal{I}}}$  such that  $\{[x], [y]\} <^{\bar{\mathcal{I}}} \{[x], [z]\}$ . That is, there are  $x_0 \in [x]$  and  $y_0 \in [y]$  such that  $\{x_0, y_0\} <^{\mathcal{I}} \{x', z'\}$  for all  $x' \in [x]$  and  $z' \in [z]$ . By the limit assumption, there are  $x_1 \in [x]$  and  $y_1 \in [y]$  such that  $\{x', y'\} \not<^{\mathcal{I}} \{x_1, y_1\}$  for all  $x' \in [x]$  and  $y' \in [y]$ . Clearly,  $y_1 \in E^{\mathcal{I}}$ . Furthermore, by the modularity condition, since  $\{x_0, y_0\} <^{\mathcal{I}} \{x_1, z\}$  and  $\{x_0, y_0\} \not<^{\mathcal{I}} \{x_1, y_1\}$ , we have  $\{x_1, y_1\} <^{\mathcal{I}} \{x_1, z\}$ . Thus,  $x_1 \in (E \Leftarrow F)^{\mathcal{I}}$  and, finally,  $x \in (E \Leftarrow F)^{\mathcal{I}}$  because  $[x] = [x_1]$ . □

## C A Tableau calculus

### C.1 Termination

*Proof (Proposition 12).* (i) We have two cases: (a) if  $\sigma^*(\Gamma', x) = x$ , it follows from the fact that the rules do not delete any formulas. Then if  $x : A$  is in  $\Gamma$ ,  $x : A$  is in  $\Gamma'$ . (b) if  $\sigma^*(\Gamma', x) = y \neq x$ , there exists a substitution chain between  $x$  and  $y$ . This substitution changes only the label to which formulas are attached. Thus, if  $x : A$  is in  $\Gamma$ , then  $y : A$  is in  $\Gamma'$ . □

*Proof (Fact 13).* We first note two that: (a) in an infinite branch there are infinitely many applications of dynamic rules (static rules alone cannot lead to an infinite derivation because of proposition 12). (b) since the number of formulas that can occur in a derivation is finite, there is a dynamic formula  $D$  to which the corresponding dynamic rule is applied infinitely often. We can then extract a subsequence  $(\Delta_n)_{n \in \mathbb{N}}$  from  $\mathbf{B}$  and a corresponding sequence of labels  $(y_k)_{k \in \mathbb{N}}$  such that for all  $k$ ,  $y_k : D \in \Delta_k$ , and  $\Delta_k$  is the node in  $\mathbf{B}$  where the corresponding dynamic rule has been applied to  $y_k : D$ .

We have also that there is a finite number of different  $\Pi_{\Delta_k}(y_k)$  (proposition 14-2). Thus, as they cannot be all different, we can find a subsequence  $(\Omega_n)_{n \in \mathbb{N}}$  of  $(\Delta_n)_{n \in \mathbb{N}}$ , together with a corresponding subsequence  $(z_n)_{n \in \mathbb{N}}$  of  $(y_n)_{n \in \mathbb{N}}$ , such that for all  $k, l$   $\Pi_{\Omega_k}(z_k) = \Pi_{\Omega_l}(z_l)$ .

Now, consider the first of such labels  $z_0$ . As the number of labels older than  $z_0$  in the branch  $\mathbf{B}$  is finite, there exists an  $i^*$  such that for all  $j > i^*$ , we have  $\sigma^*(\Omega_j, z_0) = \sigma^*(\Omega_{i^*}, z_0)$  (since, if such a  $i^*$  did not exist, there would be an infinite substitution chain where  $z_0$  is replaced by older labels, which would entail that there is an infinite number of labels older than  $z_0$  in the branch.). Let  $x^*$  be  $\sigma^*(\Omega_{i^*}, z_0)$ . We then have, by proposition 12, that  $\Pi_{\Omega_0} z_0 \subseteq \Pi_{\Omega_{i^*}}(x^*)$ , and we also have that for all  $j > i^*$ ,  $x^*$  occurs in  $\Omega_j$ . Thus we have that for all  $j > i^*$ ,  $\Pi_{\Omega_j}(z_j) \subseteq \Pi(\Omega_j)(x^*)$ .

We can finally extract the sequences  $(\Theta_n)_{n \in \mathbb{N}}$  and  $(x_n)_{n \in \mathbb{N}}$  by taking for all  $k$ :  $\Theta_k = \Omega_{i^*+k}$  and  $x_k = z_{i^*+k}$ .  $\square$

*Proof (Proposition 14).* It is easy to see that  $\text{Card}\Pi(T) \leq 2n - 1$ , since  $\Pi(T) \subseteq \text{sub}(C)$  (the rules only add subformulas of the initial concept  $C$  or they negation). And as for any tableau set  $\Gamma$  in  $T$  and any label  $x$  occurring in  $\Gamma$ ,  $\Pi_\Gamma(x) \subseteq \Pi(T)$  we obtain  $\text{Card}\Pi_\Gamma(x) \leq 2n - 1$ . The conclusion then easily follows.  $\square$

## C.2 Soundness

*Proof (Theorem 17).* Most of the cases are easy. We just present here the cases concerning the rules for  $\Leftarrow$ .

(T1 $\Leftarrow$ ) **and** (F1 $\Leftarrow$ ) they are obviously sound.

(T2 $\Leftarrow$ ) Let  $x : A \Leftarrow B$  be in  $\Gamma$ , and let  $\Gamma$  be satisfiable in  $\mathcal{I}$  under a min-mapping  $m$ . We let  $\Gamma_1 = \Gamma, z : f(x, A), z : A$  be the conclusion of the rule. Since  $m(x) \in (A \Leftarrow B)^{\mathcal{I}}$ , and by the semantics of  $\Leftarrow$ , we have  $A^{\mathcal{I}} \neq \emptyset$ . By limit assumption,  $\min_{m(x)}(A^{\mathcal{I}}) \neq \emptyset$ . Let  $z_0 \in \min_{m(x)}(A^{\mathcal{I}})$ . We can construct a function  $m'$  from  $\text{Lab}_{\Gamma_1}$  to  $\Delta^{\mathcal{I}}$  as follow:  $m'(u) = m(u)$  for all  $u \neq z$ , and  $m'(z) = z_0$ . It is then easy to check that  $m'$  is a min-mapping from  $\Gamma_1$  to  $\mathcal{I}$  and that  $\Gamma_1$  is satisfiable in  $\mathcal{I}$  under  $m'$ .

(T3 $\Leftarrow$ ) Let  $x : A \Leftarrow B, z : f(x, A), y : B$  be in  $\Gamma$ , and let  $\Gamma$  be satisfiable in  $\mathcal{I}$  under a min-mapping  $m$ . Let  $\Gamma_1 = \Gamma, \{x, z\} < \{x, y\}$  be the conclusion of the rule. Since  $m(x) \in (A \Leftarrow B)^{\mathcal{I}}, m(z) \in \min_{m(x)}(A^{\mathcal{I}})$  and  $m(y) \in B^{\mathcal{I}}$ , by the semantics of  $\Leftarrow$ ,  $\{m(x) \text{ that } m(z)\} <^{\mathcal{I}} \{m(x), m(y)\}$ .

(F2 $\Leftarrow$ ) Let  $x : \neg(A \Leftarrow B), y : A$  be in  $\Gamma$ , and let  $\Gamma$  be satisfiable in  $\mathcal{I}$  under a min-mapping  $m$ . Let  $\Gamma_1 = \Gamma, z : f(x, B), z : B$  be the conclusion of the rule. Since  $m(y) \in A^{\mathcal{I}}, A^{\mathcal{I}} \neq \emptyset$ . As  $m(x) \in \neg(A \Leftarrow B)^{\mathcal{I}}$ , by the semantics, we have that  $B^{\mathcal{I}}$  cannot be empty. By limit assumption, there exists a  $z_0 \in \min_{m(x)}(B^{\mathcal{I}})$ . We can construct a function  $m'$  from  $\text{Lab}_{\Gamma_1}$  to  $\Delta^{\mathcal{I}}$  as follow:  $m'(u) = m(u)$  for all  $u \neq z$ , and  $m'(z) = z_0$ . It is easy to check that  $m'$  is a min-mapping from  $\Gamma_1$  to  $\mathcal{I}$  and that  $\Gamma_1$  is satisfiable in  $\mathcal{I}$  under  $m'$ .

(F3 $\Leftarrow$ ) Let  $x : \neg(A \Leftarrow B), z : f(x, B), y : A$  be in  $\Gamma$ , and let  $\Gamma$  be satisfiable in  $\mathcal{I}$  under a min-mapping  $m$ . Let  $\Gamma_1 = \Gamma, \{x, y\} \not< \{x, z\}$  be the conclusion of

the rule. As  $m(x) \in \neg(A \Leftarrow B)^{\mathcal{I}}, m(z) \in \min_{m(x)}(B^{\mathcal{I}})$  and  $m(y) \in A^{\mathcal{I}}$ , we have by definition of the semantics of  $\Leftarrow$ ,  $\{m(x), m(y)\} \not\Leftarrow^{\mathcal{I}} \{m(x), m(y)\}$ .  $\square$

*Proof (Corollary 18).* If  $C$  is satisfiable, then the root node of the derivation  $\{x : C\}$  is satisfiable under a min-mapping. Since a derivation is finite (as proved in the previous section) every branch contains a finished tableau set. By the soundness of the rules, one of these tableau sets will be satisfiable.  $\square$

**Lemma 28.** *Let  $\Gamma$  be a finished tableau set, and let  $x$  and  $y$  be two labels occurring in  $\Gamma$  such that  $x$  is older than  $y$  and  $\Pi_{\Gamma}(y) \subseteq \Pi_{\Gamma}(x)$ . Then we have for any  $u, v, z$  occurring in  $\Gamma$ : if  $\{u, v\} < \{z, y\} \in \Gamma$  and it has been introduced by an application of (T3 $\Leftarrow$ ) or (tr), then  $\{u, v\} < \{z, x\} \in \Gamma$ .*

*Proof (Lemma 28).* Let the relation have the form  $\{u, v\} < \{u, y\}$  and suppose that it has been introduced by the application of (T3 $\Leftarrow$ ) to some formulas  $u : A \Leftarrow B, v : f(u, A), y : B$ . Since  $\Pi_{\Gamma}(y) \subseteq \Pi_{\Gamma}(x)$ , we have that  $x : B$  is in  $\Gamma$ , and, as  $\Gamma$  is finished, the rule (T3 $\Leftarrow$ ) has also been applied to  $u : A \Leftarrow B, v : f(u, A), x : B$ , and therefore  $\{u, v\} < \{u, x\}$  is also in  $\Gamma$ . Otherwise, the relation has been added by transitivity (tr); we then proceed by an easy induction on the number of applications of (tr) rule.  $\square$

*Proof (Fact 22).* By absurd, suppose that centering (cnt) has never been applied to  $x$  and  $y$ . We have two cases: (i)  $\{x, x\} < \{x, y\} \notin \Gamma$  or  $\{y, y\} < \{y, x\} \notin \Gamma$ : in this case, the centering rule is applicable to  $x$  and  $y$  in  $\Gamma$  (as it respects the restrictions of Definition 8). This contradicts the fact that  $\Gamma$  is finished. (ii)  $\{x, x\} < \{x, y\}$  and  $\{y, y\} < \{y, x\}$  are in  $\Gamma$ , and the centering rule was blocked by the irredundancy restriction of Definition 8. Since centering has not been applied, these formulas have been introduced by some applications of (T3 $\Leftarrow$ ) or (tr), as those are the only rules that can add this kind of formulas, apart from centering. By application of proposition 28, we have, since  $\{x, x\} < \{x, y\} \in \Gamma$ ,  $\{x, x\} < \{x, x\} \in \Gamma$ . Thus  $\Gamma$  is closed, whence it is not satisfiable, which gives a contradiction.

Since  $\Gamma$  is a descendant of  $\Delta$ , we can use the  $\sigma^*$  function defined in section 5.1 to define the mapping  $m'$ :

$$\forall x \text{ occurring in } \Delta, m'(x) = m(\sigma^*(\Gamma, x)).$$

It is easy to show that  $m'$  is indeed a min-mapping from  $\Delta$  to  $\mathcal{I}$  and that  $\Delta$  is satisfiable in  $\mathcal{I}$  under  $m'$ .  $\square$

*Proof (Fact 23).* To simplify the notations, we denote, for all label  $u$  occurring in  $\Delta$ ,  $\sigma^*(\Gamma, u)$  as  $u^*$ . Note that  $x^* = x$  and  $y^* = y$ .

**(Fact 23-1)** We have to check that the preferential relation satisfies pair-centering, modularity and asymmetry. The easy cases are preferential relations not involving  $[m'(x)]$  (or  $[m'(y)]$ ), since they are the same as in  $\mathcal{I}$ . The cases involving  $[m'(x)]$  are done in the same way as in the proof of Lemma 5.

**(Fact 23-2)**

$(\{u, v\} < \{w, z\} \in \Delta(y/x) \Rightarrow \{m''(u), m''(v)\} <^{\mathcal{I}'} \{m''(w), m''(z)\})$ : The cases where  $x$  is not involved or where the preferential relation has been introduced by a previous application of centering are easy, (as those preferential relations have not changed, neither in  $\Delta(y/x)$  nor in  $\Gamma'$ ). We shall treat the cases where  $x$  is involved.

1. The preferential relation has the form  $\{u, v\} < \{u, x\}$  and has been added by an application of the rule (T3 $\Leftarrow$ ). This relation comes either from  $\{u, v\} < \{u, x\}$  or  $\{u, v\} < \{u, y\}$  in  $\Delta$ . To ease the proof, being the other case similar, we suppose that  $\{u, v\} < \{u, x\} \in \Delta$ . Then, since this relation has been introduced by (T3 $\Leftarrow$ ), we have that  $u : A \Leftarrow B, v : f(u, A), v : A, x : B$  are in  $\Delta$ , and therefore, by proposition 12,  $u^* : A \Leftarrow B, v^* : f(u^*, A), v^* : A, x^* : B$  are in  $\Gamma$ . We note that  $y^* : B$  is in  $\Gamma$ , since (i) we cannot have  $y^* : \neg B$  in  $\Gamma$  as it would have made  $\Gamma$  unsatisfiable, by the fact that  $\Pi_\Gamma(y) \subseteq \Pi_\Gamma(x)$ . (ii) if neither  $y^* : B$  nor  $y^* : \neg B$  were in  $\Gamma$ , we could apply the rule (T1 $\Leftarrow$ ) to  $u^* : A \Leftarrow B$  and  $y$ , and thus contradict the fact that  $\Gamma$  is finished. Notice also that  $u^*$  and  $v^*$  must be different from  $x$  or  $y$  (if  $u^*$  was equal to either  $x$  or  $y$  we would have  $u^* : A \Leftarrow B, u^* : B \in \Gamma$ , which will make  $\Gamma$  unsatisfiable; similarly if  $z^*$  were equal to  $x$  or  $y$ :  $u^* : A \Leftarrow B, z^* : f(u^*, A), z^* : B$  would also be unsatisfiable). Since  $\Gamma$  is finished, the rule (T3 $\Leftarrow$ ) has also been applied to  $u^* : A \Leftarrow B, z^* : f(u^*, A), y : B$ , and so  $\{u^*, v^*\} < \{u^*, y\}$  is in  $\Gamma$ . Since  $\Gamma$  is satisfiable in  $\mathcal{I}$  this relation wrt. mapping  $m$  holds also in  $\mathcal{I}$ . Thus we have  $\{m(u^*), m(v^*)\} <^{\mathcal{I}} \{m(u^*), x\}$  and  $\{m(u^*), m(v^*)\} <^{\mathcal{I}} \{m(u^*), y\}$ . By construction of  $\mathcal{I}'$ , we obtain that  $\{[m(u^*)], [m(v^*)]\} <^{\mathcal{I}'} \{[m(u^*)], [x]\}$ , which gives the result (by definition of  $m''$ ).

2. the preferential relation has the form  $\{x, u\} < \{v, z\}$  and was previously added by an application of the rule (T3 $\Leftarrow$ ). This relation comes either from  $\{x, u\} < \{v, z\}$  or  $\{y, u\} < \{v, z\}$  in  $\Delta$ . To ease the proof, being the other case similar, we suppose that  $\{x, u\} < \{v, z\} \in \Delta$ . We then have two cases:

(i)  $x : A \Leftarrow B, u : f(x, A), z : B$  are in  $\Delta$ . In this case, we have  $v = x$  and  $z \neq x$  ( $\{x, u\} < \{x, x\}$  would have made  $\Gamma$  unsatisfiable as, due to centering and asymmetry,  $\{x, u\} \not< \{x, x\}$  is in  $\Gamma$ ). We note that  $z \neq y$  because  $\Pi_\Gamma(y) \subseteq \Pi_\Gamma(x)$ : we would have get  $x : A \Leftarrow B, y : B, x : B$  (which is unsatisfiable) in  $\Gamma$ . Now we consider  $u$ . If  $u = x$  or  $u = y$ ,  $\{x, x\} < \{x, z\} \in \Delta(y/x)$ . The proof is then easy, as it is an instance of the centering case. If  $y \neq u \neq x$ , we have that  $\{x, u^*\} < \{x, z^*\}$  is in  $\Gamma$  and then since  $\Gamma$  is satisfiable in  $\mathcal{I}$ , this relation wrt. mapping  $m$  still holds in  $\mathcal{I}$ . As  $v, z, u$  are all different from  $x$  or  $y$ , the classes  $[u], [v], [z]$  are singleton, and then we easily conclude, by definition of  $\mathcal{I}'$  that  $\{[x], [m(u^*)]\} <^{\mathcal{I}'} \{[x], [m(z^*)]\}$  which gives the result.

(ii)  $u : A \Leftarrow B, x : f(u, A), z : B$  are in  $\Delta$ . In this case we have that  $v = x$ . As in (i), we also have that  $y \neq z \neq x$  would have made  $\Gamma$  unsatisfiable. Then we consider  $u$ . if  $u = x$  or  $u = y$ ,  $\{x, x\} < \{x, z\} \in \Delta(y/x)$ .  $\{x, z\} \not< \{x, x\} \in \Delta(y/x)$ . The proof is then easy, as it is an instance of the centering case. If  $y \neq u \neq x$ , we then have that  $\{u^*, x\} < \{u^*, z^*\} \in \Gamma$  and then this relation still holds in  $\mathcal{I}$  wrt. mapping  $m$ . We conclude in the same manner as in (i).

3. the preferential relation has been introduced by an application of the transitivity rule (tr): since this rule only propagates relation that were introduced by the rule (T3 $\Leftarrow$ ), we obtain the result from the previous cases.

( $\{u, v\} \not\prec \{w, z\} \in \Delta(y/x) \Rightarrow \{m''(u), m''(v)\} \not\prec^{\mathcal{I}'} \{m''(w), m''(z)\}$ ): as above, cases where  $x$  is not involved or where this relation has been introduced by a previous application of the (asm) rule are easy. We treat the cases involving  $x$ .

1. the relation has the form  $\{u, x\} \not\prec \{u, z\}$  and has been introduced by an application of (F3 $\Leftarrow$ ). The proof is similar to the one done in the previous part: we suppose to ease the proof that  $\{x, u\} \not\prec \{v, z\} \in \Delta$ . Then we have that  $u : \neg(A \Leftarrow B), z : f(u, B), x : A$  are in  $\Delta$ . Therefore  $u^* : \neg(A \Leftarrow B), z^* : f(u, B), x : A$  are in  $\Gamma$ . As  $\Pi_{\Gamma}(y) \subseteq \Pi_{\Gamma}(x)$ , we have  $y : B \in \Gamma$ , and thus  $\{x, u^*\} \not\prec \{u^*, z^*\}$  and  $\{y, u^*\} \not\prec \{u^*, z^*\}$  are in  $\Gamma$ . We conclude by the satisfiability of  $\Gamma$  in  $\mathcal{I}$  under  $m$ , and by the definition of  $\mathcal{I}'$ .

2. the relation has the form  $\{v, z\} \not\prec \{u, x\}$  and has been introduced by an application of (F3 $\Leftarrow$ ). The proof is similar to the above one. We suppose to ease the proof that  $\{v, z\} \not\prec \{u, x\} \in \Delta$ . We then have two cases:

(i)  $x : \neg(A \Leftarrow B), u : f(x, B), z : A$  are in  $\Delta$ : in this case  $v = x$ . We then consider label  $u$ . if  $u = x$  or  $u = y$ ,  $\{x, z\} \not\prec \{x, x\} \in \Delta(y/x)$ . The proof is then easy, as it is an instance of the centering case. If  $y \neq u \neq x$ , then  $y \neq z \neq x$  (as  $\Gamma$  would then have been unsatisfiable due to (cnt) and (asm)). Thus we have that  $\{x^*, z^*\} \not\prec \{x^*, u^*\}$  is in  $\Gamma$ . We conclude by the satisfiability of  $\Gamma$  in  $\mathcal{I}$  under  $m$ , and by the definition of  $\mathcal{I}'$ .

(ii)  $u : \neg(A \Leftarrow B), x : f(u, B), z : A$  are in  $\Delta$ : in this case  $v = u$ . We consider  $u$ . If  $u = x$  or  $u = y$ ,  $\{x, z\} \not\prec \{x, x\} \in \Delta(y/x)$ . The proof is then easy, as it is an instance of the centering case. If  $y \neq u \neq x$ , then we also have that  $y \neq z \neq x$ . We can then conclude in the same way as (i).

( $z : f(u, A) \in \Gamma \Rightarrow m''(z) \in \min_{m''(u)}(A^{\mathcal{I}'})$ ): The proof is easy.

**(Fact 23-3)** We prove that if  $u : D \in \Delta(y/x)$ , then  $m''(u) \in D^{\mathcal{I}'}$  by induction on the structure of the formula. The booleans cases are easy. We show here the case of concept similarity formulas.

( $u : A \Leftarrow B \in \Delta(y/x) \Rightarrow m''(u) \in (A \Leftarrow B)^{\mathcal{I}'}$ ): 1.  $u = x$ . In  $\Delta$ , we have either  $x : A \Leftarrow B$  or  $y : A \Leftarrow B$ . Note that the latter implies, since  $\Pi_{\Gamma}(y) \subseteq \Pi_{\Gamma}(x)$  that  $x : A \Leftarrow B$  is in  $\Gamma$ . Then we necessarily have that  $x : A \Leftarrow B$  is in  $\Gamma$ . This implies that  $x : \neg B$  and  $y : \neg B$  are in  $\Gamma$  (due to the rule (T1 $\Leftarrow$ ) and the fact  $x : A \Leftarrow B, x : B$  is unsatisfiable). As  $\Gamma$  is satisfiable in  $\mathcal{I}$  under  $m$ , there exists a  $z_0 \in A^{\mathcal{I}}$  such that for all  $v \in B^{\mathcal{I}}$ ,  $\{m(x), z_0\} <^{\mathcal{I}} \{m(x), v\}$ . We also note that by induction hypothesis  $m(y)$  and  $m(x)$  are not in  $B^{\mathcal{I}}$  ( $x : \neg B$  and  $y : \neg B$  are in  $\Gamma$ ) Therefore, by definition of  $\mathcal{I}'$ , we obtain that for all  $[v] \in B^{M'}$ ,  $\{[m(x)], [z_0]\} <^{\mathcal{I}'} \{[m(x)], [v]\}$ . Thus,  $[m(x)] \in (A \Leftarrow B)^{\mathcal{I}'}$ .

2.  $u \neq x$ . Then  $u : A \Leftarrow B$  is in  $\Delta$ , and thus  $u^* : A \Leftarrow B \in \Gamma$ . As  $\Gamma$  is satisfiable in  $\mathcal{I}$  under  $m$ , there exists a  $z_0 \in \Delta^{\mathcal{I}}$  such that  $z_0 \in A^{\mathcal{I}}$  and for all  $v \in B^{\mathcal{I}}$ ,  $\{m(u^*), z_0\} <^{\mathcal{I}} \{m(u^*), v\}$ . Since  $\Gamma$  is finished, the rule (T1 $\Leftarrow$ ) has been applied to  $y$ . We then have several cases:

(i)  $y : B \in \Gamma$ , and so is  $x : B$ . Then we have that  $m(y) \neq z_0 \neq m(x)$  (as it would give a contradiction with the asymmetry of  $<^{\mathcal{I}}$ ). As  $m(x) \in B^{\mathcal{I}}$  and  $y \in B^{\mathcal{I}}$ , we

have  $\{m(u^*), z_0\} <^{\mathcal{I}} \{m(u^*), m(x)\}$  and  $\{m(u^*), z_0\} <^{\mathcal{I}} \{m(u^*), m(y)\}$ . By definition of  $\mathcal{I}'$ , we have  $\{[m(u^*)], [z_0]\} <^{\mathcal{I}'} \{[m(u^*)], [m(x)]\}$ . The elements different from  $m(x)$  or  $m(y)$  in  $\Delta^{\mathcal{I}}$  being unaffected by the partial filtration, we have, for all  $[z] \in \mathbf{B}^{\mathcal{I}'}$ ,  $\{[m(u^*)], [z_0]\} <^{\mathcal{I}'} \{[m(u^*)], [z]\}$ . Thus  $[m(u^*)] \in (A \Leftarrow B)^{\mathcal{I}'}$ .

(ii)  $y : \neg B \in \Gamma$ , and so is  $x : \neg B$ . Since  $\Gamma$  is finished, the rule (T2 $\Leftarrow$ ) has been applied to  $u^* : (A \Leftarrow B)$ , hence there exists a label  $z$  such that  $z : f(u^*, A), z : A \in \Gamma$ . Thus, as  $\Gamma$  is satisfiable in  $\mathcal{I}$  under  $m$ , we have that for all  $v \in B^{\mathcal{I}}$ ,  $\{m(u), m(z)\} <^{\mathcal{I}} \{m(u), v\}$ . If  $y \neq z \neq x$ , we conclude easily by the construction of the partially filtrated model (and since neither  $m(x)$  nor  $m(y)$  are in  $B^{\mathcal{I}}$ ). If  $y = z$ , by the fact that  $\Pi_{\Gamma}(y) \subseteq \Pi_{\Gamma}(x)$ ,  $x : A$  is also in  $\Gamma$ , and thus  $[m(x)] \in A^{\mathcal{I}'}$  (by definition of  $\mathcal{I}'$ ). Then we have for all  $[v] \in B^{\mathcal{I}'}$ ,  $\{[m(u^*)], [m(x)]\} <^{\mathcal{I}'} \{[m(u^*)], [v]\}$ . The case where  $z = x$  is similar. And thus  $[m(u^*)] \in (A \Leftarrow B)^{\mathcal{I}'}$ .

( $u : \neg(A \Leftarrow B) \Rightarrow m''(u) \in \neg(A \Leftarrow B)$ ) 1.  $u = x$ . we have either  $x : \neg(A \Leftarrow B)$  or  $y : \neg(A \Leftarrow B)$ . Note that the latter implies, since  $\Pi_{\Gamma}(y) \subseteq \Pi_{\Gamma}(x)$  that  $x : \neg(A \Leftarrow B)$  is in  $\Gamma$ . Thus we necessarily have that  $x : \neg(A \Leftarrow B)$  is in  $\Gamma$ .  $\Gamma$  is satisfiable in  $\mathcal{I}$  under  $m$ . If  $A^{\mathcal{I}}$  is empty: we can conclude immediately. If it is not, then there exists a  $z_0$  such that  $z_0 \in \min_{m(x)}(B^{\mathcal{I}})$  and for all  $v \in A^{\mathcal{I}}$ ,  $\{m(x), v\} \not<^{\mathcal{I}} \{m(x), z_0\}$ . Not we consider two cases:

(i)  $m(x) \in B^{\mathcal{I}}$ : we have that  $z_0 = m(x)$  (as in this case  $\min_{m(x)}(B^{\mathcal{I}}) = x$ ). By definition of  $\mathcal{I}'$ , we have that  $[m(x)] \in B^{\mathcal{I}'}$ . As the relation  $<^{\mathcal{I}'}$  satisfies the pair-centering property, we have easily that  $[m(x)] \notin (A \Leftarrow B)^{\mathcal{I}'}$ .

(ii)  $m(x) \notin B^{\mathcal{I}}$ : since  $\Gamma$  is finished, the rule (F1 $\Leftarrow$ ) has been applied to  $y$ . If  $y : A$  is in  $\Gamma$ ,  $x : A$  is. Then we would have  $m(x) \notin B^{\mathcal{I}}, m(x) \in A^{\mathcal{I}}$  which contradict  $m(x) \in \neg(A \Leftarrow B)^{\mathcal{I}}$ . Thus we have  $y : \neg A$  and  $x : \neg A$  in  $\Gamma$ . Since  $A^{\mathcal{I}}$  is not empty, the rule (F2 $\Leftarrow$ ) has been applied, thus there exists a  $z$  in  $\Gamma$  such that  $z : f(x, B), z : B$  is in  $\Gamma$ . Notice that  $z \neq y$  (as if  $z = y$ , we would have  $x : B$  in  $\Gamma$  which gives a contradiction). We then let  $z_0 = m(z)$ . Since  $[m(x)] \notin A^{\mathcal{I}}$ , we easily conclude that there exists a  $[z_0] \in B^{\mathcal{I}}$  such that for all  $[v] \in A^{\mathcal{I}}$ ,  $\{[m(x)], [v]\} \not<^{\mathcal{I}'} \{[m(x)], [z_0]\}$ , and so that  $[x] \in \neg(A \Leftarrow B)^{\mathcal{I}'}$ .

2.  $u \neq x$ . Then we have  $u^* : \neg(A \Leftarrow B)$  in  $\Gamma$ . Again, if  $A^{\mathcal{I}} = \emptyset$ , the case is trivial. If  $A^{\mathcal{I}} \neq \emptyset$ , then the rule (F2 $\Leftarrow$ ) has been applied, thus there exists a  $z$  in  $\Gamma$  such that  $z : f(x, B), z : B$  are in  $\Gamma$ , so that for all  $v \in A^{\mathcal{I}}$ ,  $\{m(u), v\} \not<^{\mathcal{I}} \{m(u), m(z)\}$ . The rule (F1 $\Leftarrow$ ) has also been applied to  $y$ , we then have several cases:

(i)  $y : \neg A$  is in  $\Gamma$ , and so is  $x : \neg A$ . If  $z = y$ , we have that  $x : B$  is in  $\Gamma$ , and thus  $[m(x)] \in B^{\mathcal{I}'}$ . The conclusion easily follows. The same if  $z = x$ . If  $x \neq z \neq y$ , the result is also easy, since the preferential relation does not involve  $x$  (as  $x : \neg A$  is in  $\Gamma$ ). (ii)  $y : A$  is in  $\Gamma$ , and so is  $x : A$ . If  $z = y$ , we have  $x : B$  in  $\Gamma$ , and then  $[x] \in B^{\mathcal{I}}$ . The same if  $x = z$ . In this case, we can conclude easily. If  $x \neq z \neq y$ , due to the rule (F3 $\Leftarrow$ ), we have that  $\{u, x\} \not< \{u, z\}$  and  $\{u, x\} \not< \{u, z\}$  are in  $\Gamma$ , so that  $\{m(u), m(x)\} \not<^{\mathcal{I}} \{m(u), m(z)\}$   $\{m(u), m(y)\} \not<^{\mathcal{I}} \{m(u), m(z)\}$ , and the conclusion easily follows.  $\square$

### C.3 Completeness

To ease the proofs, we introduce the following definition:

**Definition 29 (Saturated tableau set).** A tableau set  $\Gamma$  is saturated if (i) if  $x : \neg\neg A \in \Gamma$ , then  $x : A \in \Gamma$ . (ii) if  $x : A \sqcap B \in \Gamma$  (resp.  $x : \neg(A \sqcup B) \in \Gamma$ ), then  $x : A$  and  $x : B$  (resp.  $x : \neg A$  and  $x : \neg B$ ) are in  $\Gamma$ . (iii) if  $x : A \sqcup B \in \Gamma$  (resp.  $x : \neg(A \sqcap B) \in \Gamma$ ), then either  $x : A$  (resp.  $x : \neg A$ ) or  $x : B$  (resp.  $x : \neg B$ ) is in  $\Gamma$ . (iv) if  $x : A \Leftarrow B \in \Gamma$  (resp.  $x : \neg(A \Leftarrow B) \in \Gamma$ ), then for all  $y$  occurring in  $\Gamma$ , either  $y : B$  (resp.  $y : A$ ) or  $y : \neg B$  (resp.  $y : \neg A$ ) is in  $\Gamma$ . (v) if  $x : A \Leftarrow B \in \Gamma$ , then there exists a  $z$  such that  $z : f(x, A), z : A$  are in  $\Gamma$ , and for all  $y$  occurring in  $\Gamma$ , if  $y : B \in \Gamma$  then  $\{x, z\} < \{x, y\}$  is in  $\Gamma$ . (vi) if  $x : \neg(A \Leftarrow B) \in \Gamma$ , then either  $y : \neg A \in \Gamma$  for all  $y$  occurring in  $\Gamma$ , or there exist a  $z$  such that  $z : f(x, B), z : B$  are in  $\Gamma$  and  $\{x, y\} \not\prec \{x, z\} \in \Gamma$  for all  $y$  such that  $y : A \in \Gamma$ . (vii) for all  $x$  and  $y$  occurring in  $\Gamma$ , if  $x \neq y$  then  $\{x, x\} < \{x, y\}$  and  $\{y, y\} < \{y, x\}$  are in  $\Gamma$ . (viii) if  $\{x, y\} < \{u, v\} \in \Gamma$ , then  $\{u, v\} \not\prec \{x, y\} \in \Gamma$ . (ix) if  $\{x, y\} \not\prec \{v, w\}$  and  $\{v, w\} \not\prec \{t, u\}$  are in  $\Gamma$ , then  $\{x, y\} \not\prec \{t, u\}$  is in  $\Gamma$ . (x) if  $\{x, y\} < \{v, w\}$  and  $\{v, w\} < \{t, u\}$  are in  $\Gamma$ , then  $\{x, y\} < \{t, u\}$  is in  $\Gamma$ .

*Remark 30.* Saturation conditions ix and x implies that the relations  $<$  and  $\not\prec$  in  $\Gamma$  are transitive.

**Lemma 31.** Any good tableau set is saturated.

*Proof (Lemma 31).* Let  $\Gamma$  be a good tableau set. We have that  $\Gamma$  is finished, and that no labels is blocked by the subset blocking of Definition 8. Then all rules are blocked by the irredundancy restrictions, which easily leads to the saturation of  $\Gamma$ .  $\square$

We will also use the following properties of the relations  $\not\prec$  and  $<_e$ :

- FACT 32.** 1. If there exists  $x_1, y_1, x_2, y_2, \dots, x_n, y_n$  such that  $\{x_1, y_1\} \not\prec \{x_2, y_2\}, \{x_2, y_2\} \not\prec \{x_3, y_3\} \not\prec \dots \not\prec \{x_{n-1}, y_{n-1}\} \not\prec \{x_n, y_n\} \not\prec \{x_1, y_1\}$ , then  $\{x_1, y_1\} =_{\not\prec} \{x_2, y_2\} =_{\not\prec} \dots =_{\not\prec} \{x_n, y_n\}$ .
2. If  $\{x, y\} < \{u, v\} \in \Gamma$ , then  $[\{x, y\}]_{\not\prec} \neq [\{u, v\}]_{\not\prec}$ .
3. If  $\{x, y\} < \{u, v\} \in \Gamma$ , then  $[\{x, y\}]_{\not\prec} <_e [\{u, v\}]_{\not\prec}$ .

*Proof (Fact: 32).*

1. Consider for instance the pairs  $\{x_1, y_1\}$  and  $\{x_2, y_2\}$ . We have  $\{x_1, y_1\} \not\prec \{x_2, y_2\}$  in  $\Gamma$ . We also have  $\{x_2, y_2\} \not\prec \dots \not\prec \{x_{n-1}, y_{n-1}\} \not\prec \{x_n, y_n\} \not\prec \{x_1, y_1\}$ . By remark 30, we can conclude that  $\{x_2, y_2\} \not\prec \{x_1, y_1\}$  in  $\Gamma$ . Then we have, by definition of  $=_{\not\prec}$  that  $\{x_1, y_1\} =_{\not\prec} \{x_2, y_2\}$ . We obtain the result by applying this reasoning to all the other pairs in the sequence.
2. As  $\Gamma$  is good (and then open), we have that  $\{x, y\} \not\prec \{u, v\} \notin \Gamma$ . We conclude by definition of  $=_{\not\prec}$ .
3. By the previous Fact, we have that  $[\{x, y\}]_{\not\prec} <_e [\{u, v\}]_{\not\prec}$ , and that  $\{u, v\} \not\prec \{x, y\} \notin \Gamma$  (it would contradict the fact that  $\Gamma$  is open.). By saturation condition viii, we have that  $\{u, v\} \not\prec \{x, y\}$  is in  $\Gamma$ . We conclude by the definition of  $<_e$ .

□

We are now ready to prove all the other Facts:

*Proof (Fact 25).*

**Irreflexivity** Follows easily from the definition of  $=_{\neq}$ , and from the definition of  $<_e$ .

**Asymmetry** By absurd, suppose that  $[\{x, y\}]_{\neq} <_e [\{u, v\}]_{\neq}$  and  $[\{u, v\}]_{\neq} <_e [\{x, y\}]_{\neq}$ . Since  $<_e$  is irreflexive,  $[\{x, y\}]_{\neq} \neq [\{u, v\}]_{\neq}$ . By definition of  $<_e$ , there exists  $\{x_1, y_1\}, \{x_2, y_2\}$  in  $[\{x, y\}]_{\neq}$  and  $\{u_1, v_1\}, \{u_2, v_2\}$  in  $[\{u, v\}]_{\neq}$  such that  $\{u_1, v_1\} \not\prec \{x_1, y_1\}$  and  $\{x_2, y_2\} \not\prec \{u_2, v_2\}$  are in  $\Gamma$ . We also have by definition of  $=_{\neq}$  that  $\{x_1, y_1\} \neq \{x_2, y_2\}$  and  $\{u_1, v_1\} \not\prec \{u_2, v_2\}$ . We then have a cycle, and by application of Fact 32-1, we obtain that  $[\{x, y\}]_{\neq} = [\{u, v\}]_{\neq}$  which gives the contradiction.

**Transitivity** Let  $[\{x, y\}]_{\neq} <_e [\{u, v\}]_{\neq}$  and  $[\{u, v\}]_{\neq} <_e [\{w, z\}]_{\neq}$ . By irreflexivity of  $<_e$  we have that  $[\{x, y\}]_{\neq} \neq [\{u, v\}]_{\neq}$  and  $[\{u, v\}]_{\neq} \neq [\{w, z\}]_{\neq}$ . By asymmetry we also obtain  $[\{x, y\}]_{\neq} \neq [\{w, z\}]_{\neq}$ . There exists  $\{x_1, y_1\} \in [\{x, y\}]_{\neq}$ ,  $\{u_1, v_1\}, \{u_2, v_2\} \in [\{u, v\}]_{\neq}$  and  $\{w_1, z_1\} \in [\{w, z\}]_{\neq}$  such that  $\{u_1, v_1\} \not\prec \{x_1, y_1\}$ ,  $\{w_1, z_1\} \not\prec \{u_2, v_2\}$  are in  $\Gamma$ . By definition of  $=_{\neq}$ , we also have  $\{u_2, v_2\} \not\prec \{u_1, v_1\}$  in  $\Gamma$  (remark 30). Thus we have  $\{w_1, z_1\} \not\prec \{x_1, y_1\}$  in  $\Gamma$ . By applying the same reasoning as in the asymmetry proof, and since  $[\{x, y\}]_{\neq} \neq [\{w, z\}]_{\neq}$ , we can conclude that  $\{x_1, y_1\} \not\prec \{w_1, z_1\}$  is not in  $\Gamma$  and therefore we have  $[\{x, y\}]_{\neq} <_e [\{w, z\}]_{\neq}$  by definition of  $<_e$ .

□

*Proof (Fact 26).* We will first prove that  $\mathcal{I}_\Gamma$  is a  $\mathcal{CSL}$ -pair model. We have to check that the relation  $<^{\mathcal{I}_\Gamma}$  satisfies the properties of Definition 2.

**Pair-centering** Let  $x, y$  be two labels occurring in  $\Gamma$ . If  $x = y$  we conclude immediately. If  $x \neq y$ , we have that  $\{x, x\} < \{x, y\}$  is in  $\Gamma$  (saturation condition vii). By Fact 32-3 we have  $[\{x, x\}]_{\neq} <_e [\{x, y\}]_{\neq}$  and therefore  $[\{x, x\}]_{\neq} <^* [\{x, y\}]_{\neq}$ . We then conclude that  $\{x, x\} <^{\mathcal{I}_\Gamma} \{x, y\}$ .

**Modularity** Comes easily from the fact that  $<^*$  is a total (strict) order over the equivalence classes.

**Asymetry** Comes easily from the fact that  $<^*$  is a total (strict) order over the equivalence classes.

**Limit assumption** Due to the finiteness of every derivation under termination restrictions,  $\Gamma$  is finite and hence contains only a finite number of labels.  $\mathcal{I}_\Gamma$  is then a finite model, and trivially satisfies the limit assumption.

We now show that the identity mapping is a  $\mathcal{CSL}$ -mapping from  $\Gamma$  to  $\mathcal{I}_\Gamma$ .

( $\{x, y\} < \{u, v\} \in \Gamma \Rightarrow \{x, y\} <^{\mathcal{I}_\Gamma} \{u, v\}$ ) By Fact 32-3 we have that  $[\{x, y\}]_{\neq} <_e [\{u, v\}]_{\neq}$  and thus that  $[\{x, y\}]_{\neq} <^* [\{u, v\}]_{\neq}$ . We then conclude that  $\{x, y\} <^{\mathcal{I}_\Gamma} \{u, v\}$ .

( $\{x, y\} \not\prec \{u, v\} \in \Gamma \Rightarrow \{x, y\} \not\prec^{\mathcal{I}_\Gamma} \{u, v\}$ ) We have two cases: either  $\{u, v\} \not\prec \{x, y\}$  and hence  $[\{x, y\}]_{\neq} = [\{u, v\}]_{\neq}$  and we then conclude immediately, or

$\{u, v\} \not\prec \{x, y\}$  is not in  $\Gamma$ . Then we have that  $[\{x, y\}]_{\not\prec} \neq [\{u, v\}]_{\not\prec}$ , and by definition of  $<_e$ ,  $[\{u, v\}]_{\not\prec} <_e [\{x, y\}]_{\not\prec}$ . Therefore  $[\{u, v\}]_{\not\prec} <^* [\{x, y\}]_{\not\prec}$  and  $\{u, v\} <^{\mathcal{I}_r} \{x, y\}$ .  $<^{\mathcal{I}_r}$  being asymmetric, we obtain  $\{x, y\} \not\prec^{\mathcal{I}_r} \{u, z\}$ .

We are then ready to prove that  $\Gamma$  is satisfiable in  $\mathcal{I}_r$  under the identity mapping. This is done by induction on the structure of a formula  $D$ , and comes from the fact that  $\Gamma$  is saturated (Lemma 31). Most of the cases are easy. We only show here the cases involving  $\Leftarrow$ :

- ( $x : A \Leftarrow B \in \Gamma \Rightarrow x \in (A \Leftarrow B)^{\mathcal{I}_r}$ ) Since  $\Gamma$  is saturated, we notice that for all  $y$ , if  $y \in B^{\mathcal{I}_r}$ , then  $x : B \in \Gamma$  (saturation condition iv and induction hypothesis). By saturation condition v, there exists a label  $z$  such that  $z : f(x, A), z : A$  are in  $\Gamma$  and for all  $y$  such that  $y : B \in \Gamma$ ,  $\{x, z\} < \{x, y\}$  is in  $\Gamma$ . We then conclude by induction hypothesis.
- ( $x : \neg(A \Leftarrow B) \in \Gamma \Rightarrow x \in \neg(A \Leftarrow B)^{\mathcal{I}_r}$ )  $\Gamma$  being saturated, we notice that for all  $y$ , if  $y \in A^{\mathcal{I}_r}$ , then  $x : A \in \Gamma$  (saturation condition iv and induction hypothesis). Then by saturation condition vi, we have that either (a) there are no labels  $y$  such that  $y : A$  is in  $\Gamma$ . By induction hypothesis,  $A^{\mathcal{I}} = \emptyset$  and we conclude immediately. (b) there exists a  $z$  such that  $z : f(x, B), z : B$  and for all  $y$ , if  $y : A \in \Gamma$  then  $\{x, y\} \not\prec \{x, z\} \in \Gamma$ . We conclude again by induction hypothesis.

□